Coupled oscillators approach to data analysis

Basic theory

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Basic theory

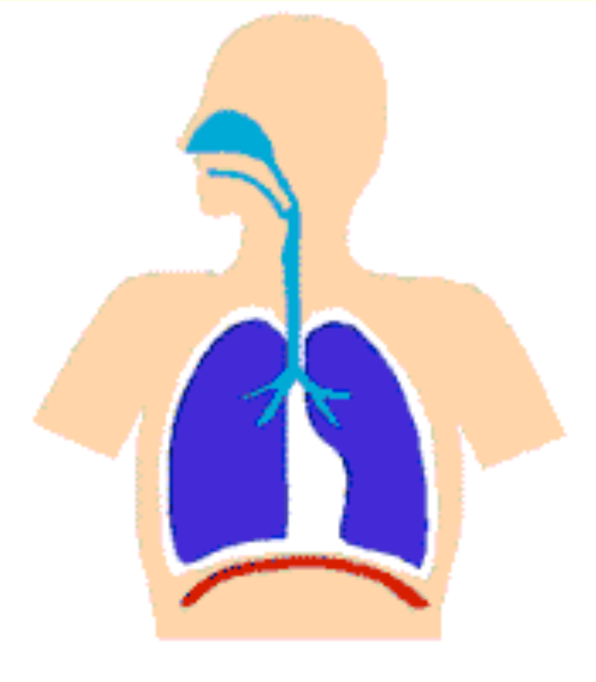
Synchronization:
A universal mechanism for adjustment of rhythms of nonlinear systems

An introductory remark

- The notion of synchrony/synchronization: understood differently in different branches of science
- This presentation: a physicist's viewpoint
- Classical physics: no quantum and relativistic effects
- Subject of intensive research: Physical Review E, October 2016: two times in the title, four times in the abstract
- Most likely: the oldest scientifically described nonlinear effect!

Synchronization: what is it about?

1 It is about **oscillatory objects**



Animated images: www.netanimations.net

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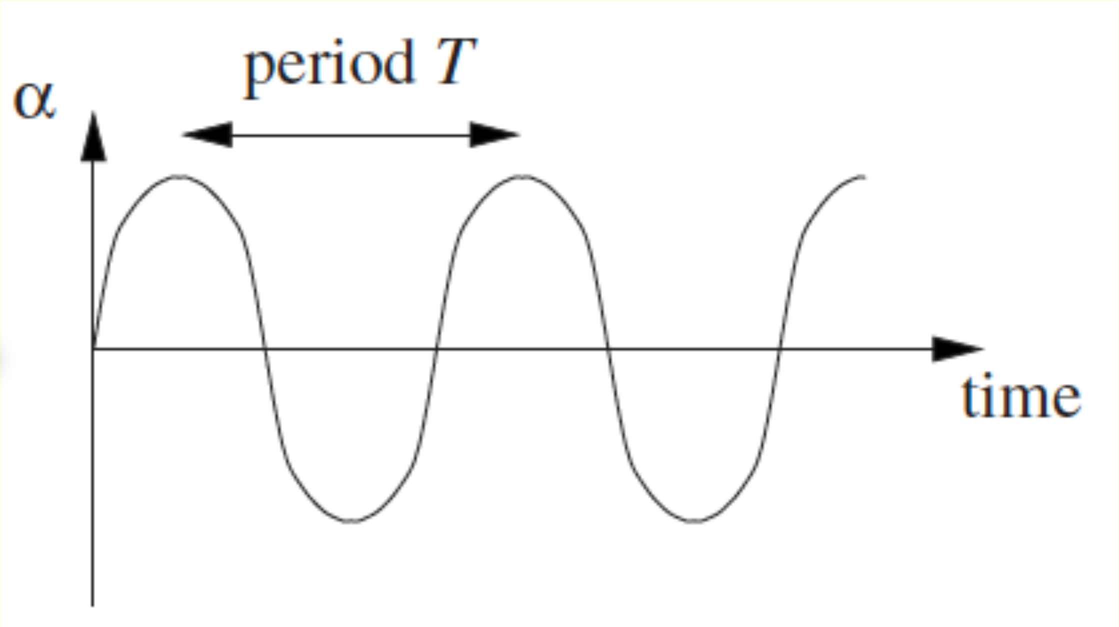

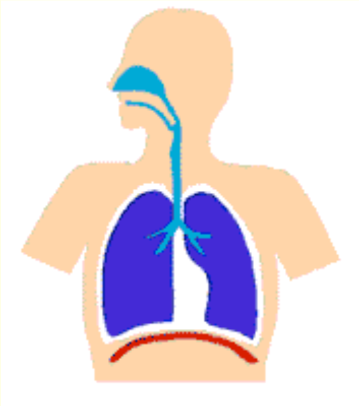

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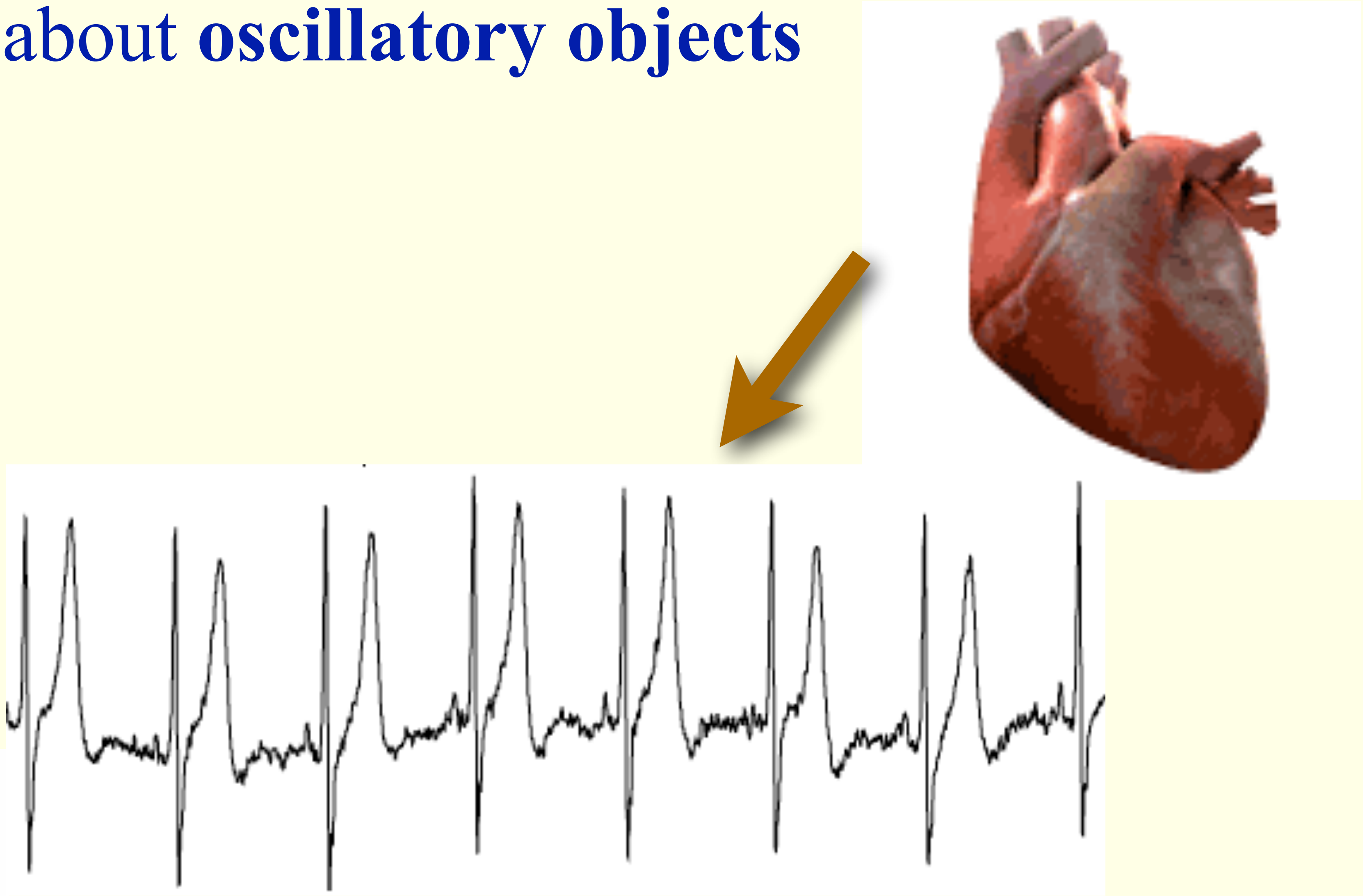

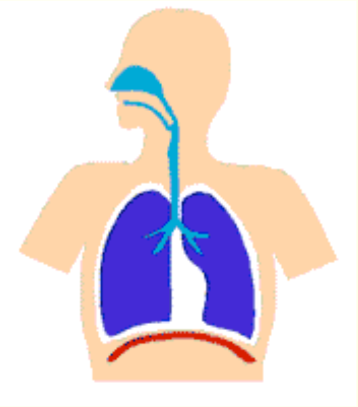

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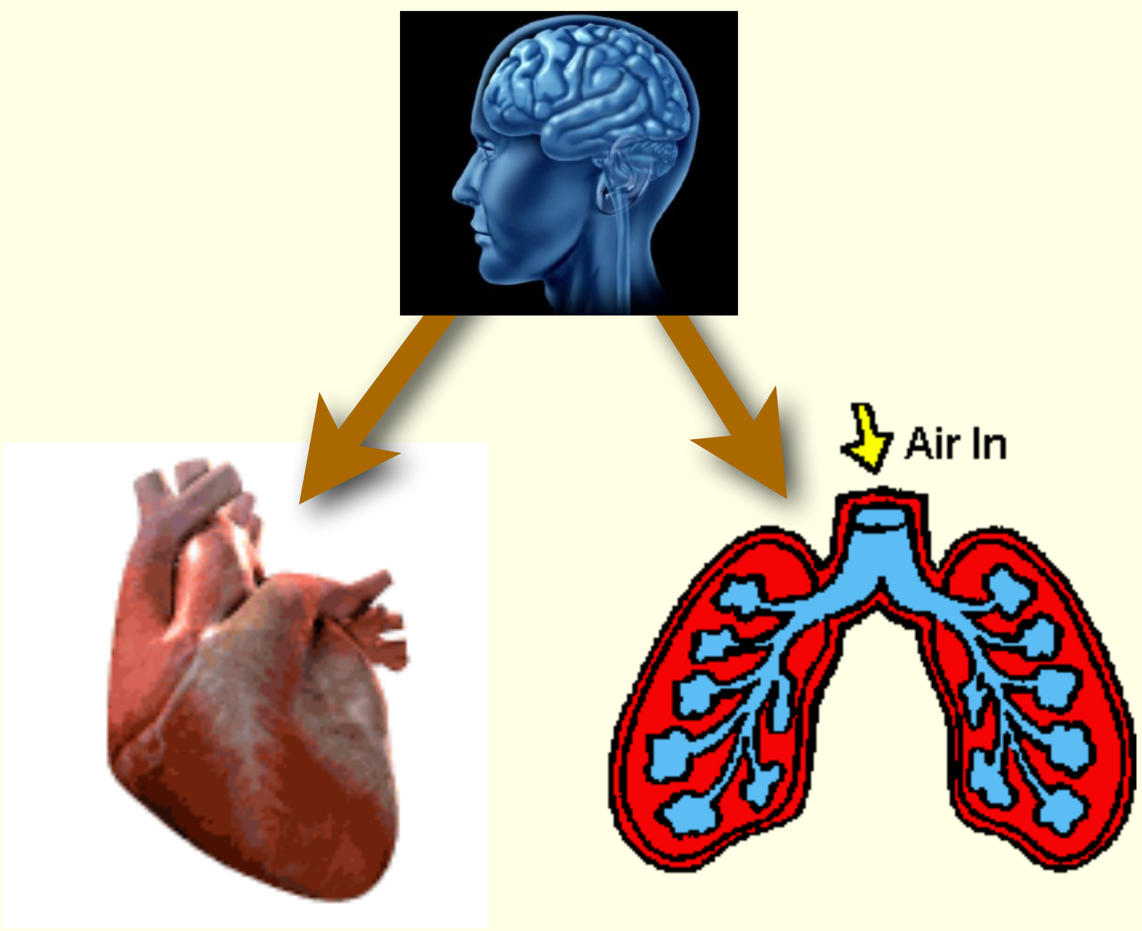
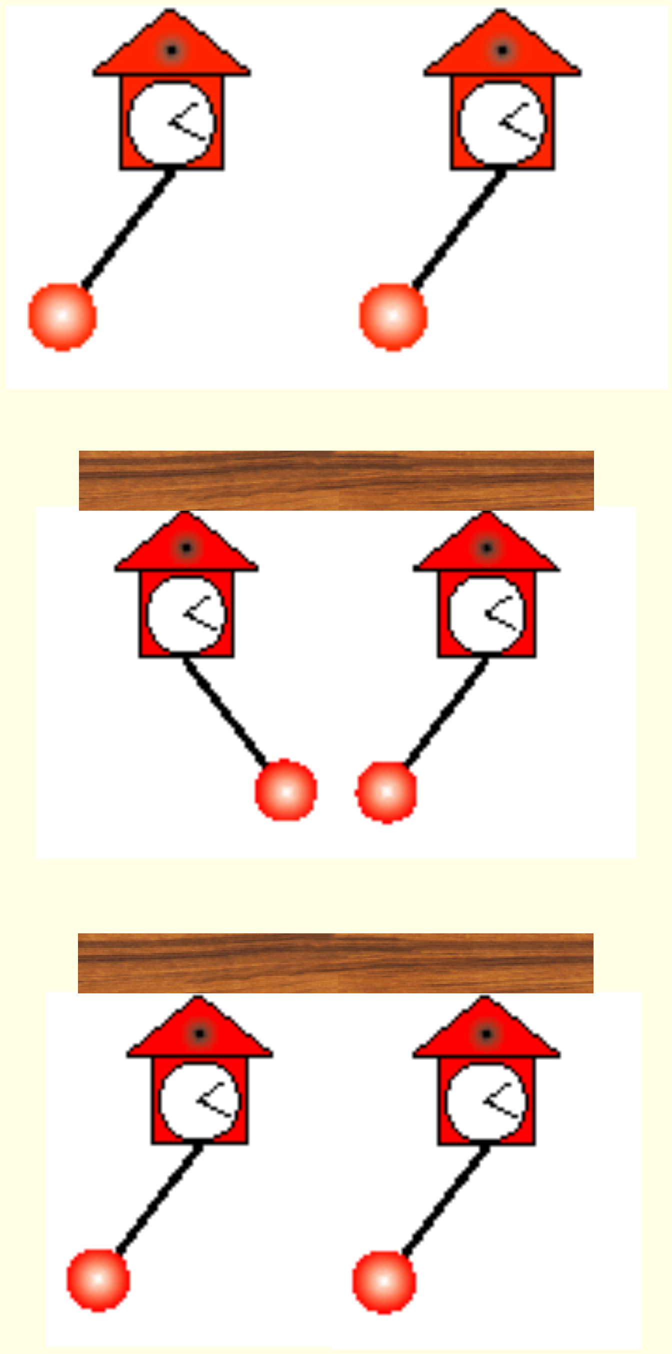
Synchronization: what is it about?

1 It is about **oscillatory objects**

2 Its about their **weak interaction**





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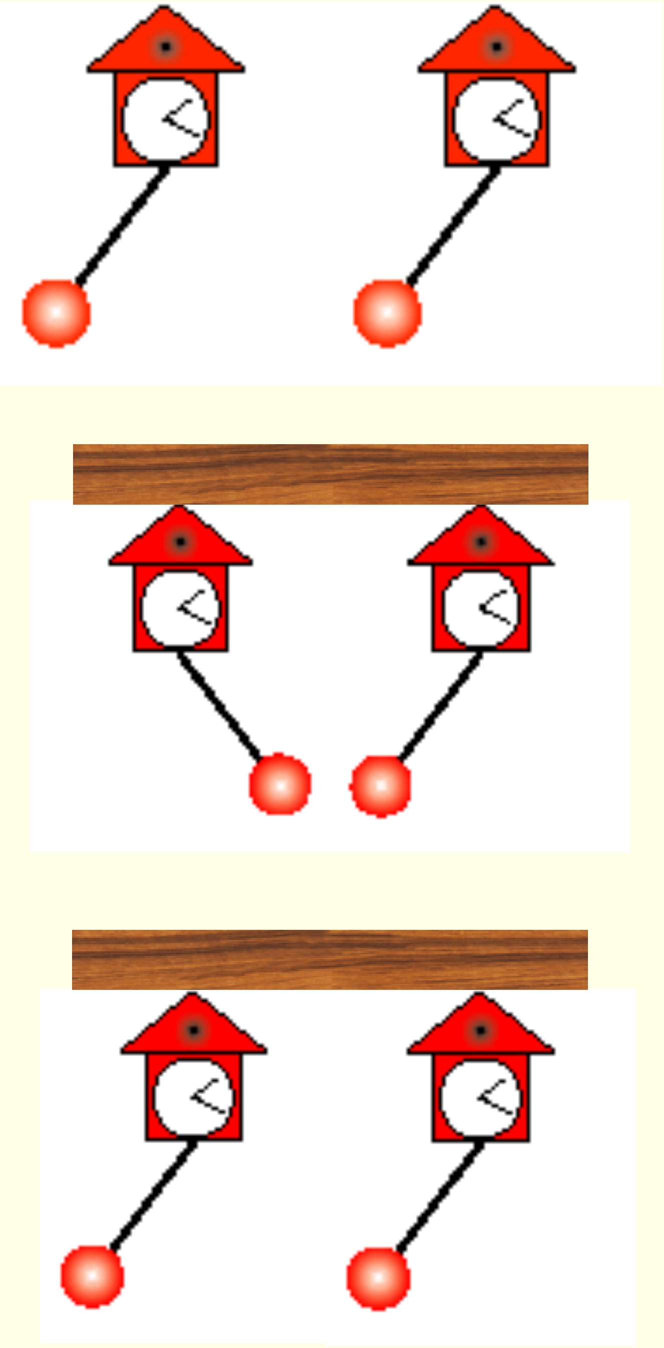
Synchronization: what it is?

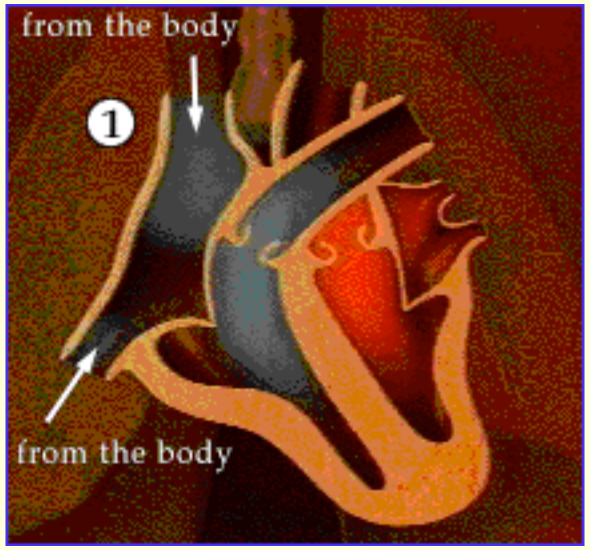
1 It is about **oscillatory objects**

2 Its about their **weak interaction**

Synchronization is adjustment of rhythms of active (self-sustained) **oscillatory objects** due to their **weak interaction**



Self-sustained oscillators



Active oscillators

Biology: systems generating **endogenous** rhythms

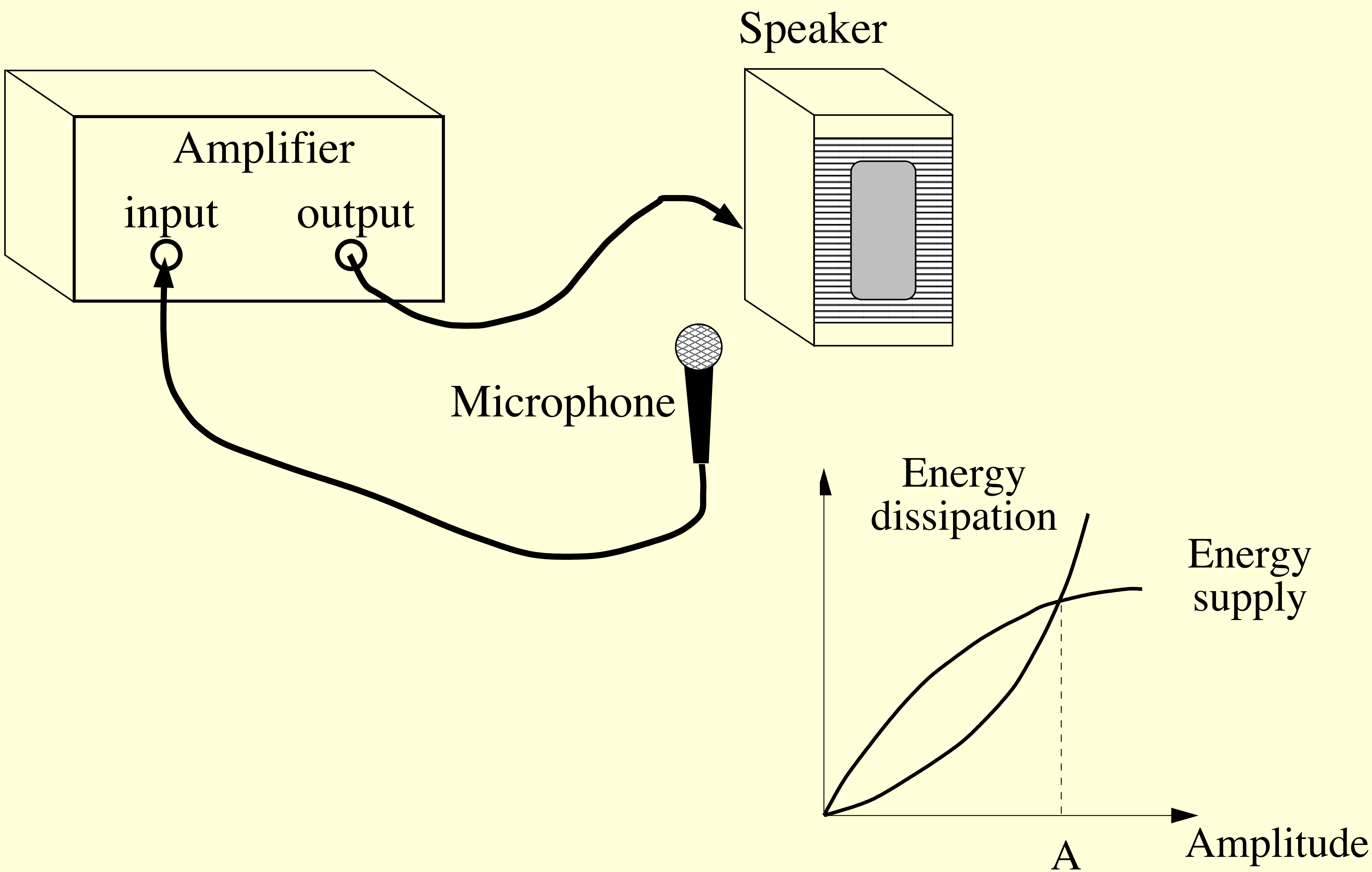
Systems of this class:

1. generate stationary oscillations without periodic forces
2. are dissipative nonlinear systems

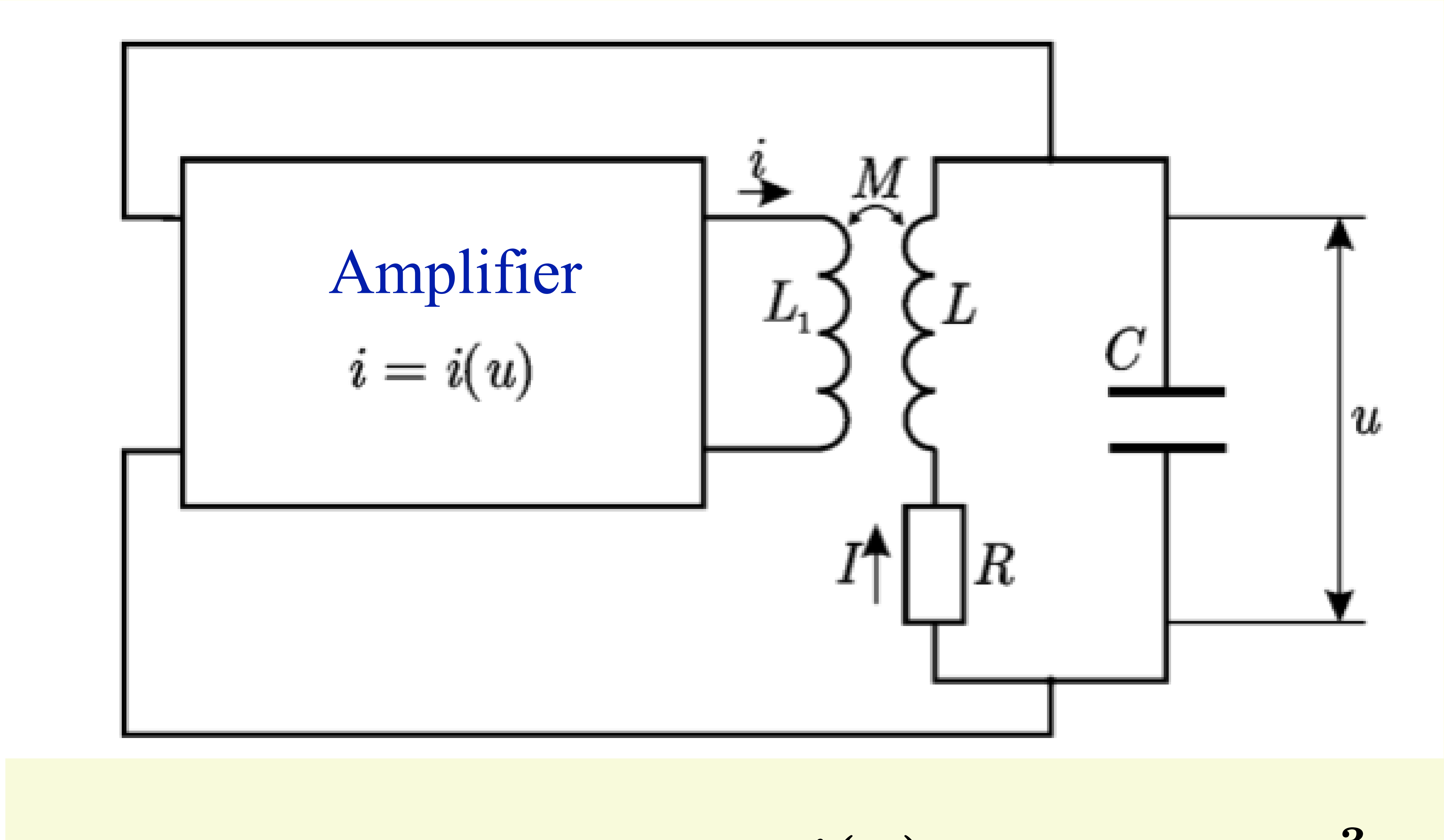
3. are described by autonomous differential equations
4. are represented by a limit cycle in the phase space

Synchronization is possible for self-sustained oscillators only!

Self-sustained oscillators: example I



Paradigmatic model: van der Pol equation



Kirchhoff law + approximation $i(u) = g_0 u - g_1 u^3$

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega^2 x = 0$$

Limit cycle

Consider general N-dimensional ($N \geq 2$) self-sustained oscillator

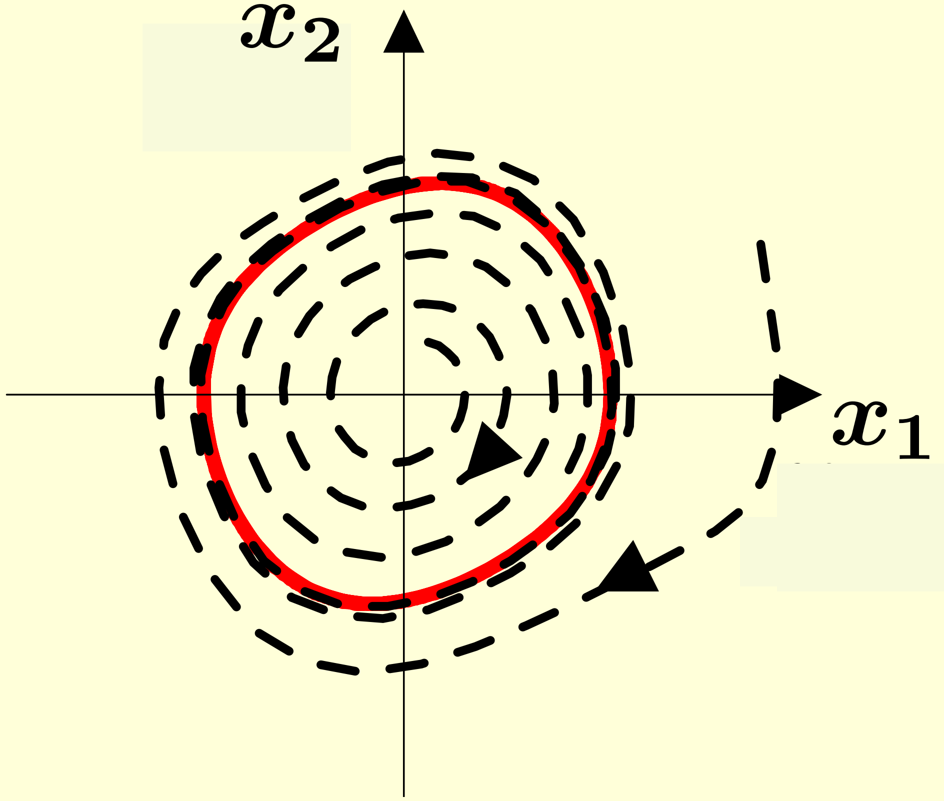
$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) \;, \mathbf{x} = (x_1, x_2, \ldots, x_N)$$

Suppose it has a stable periodic solution

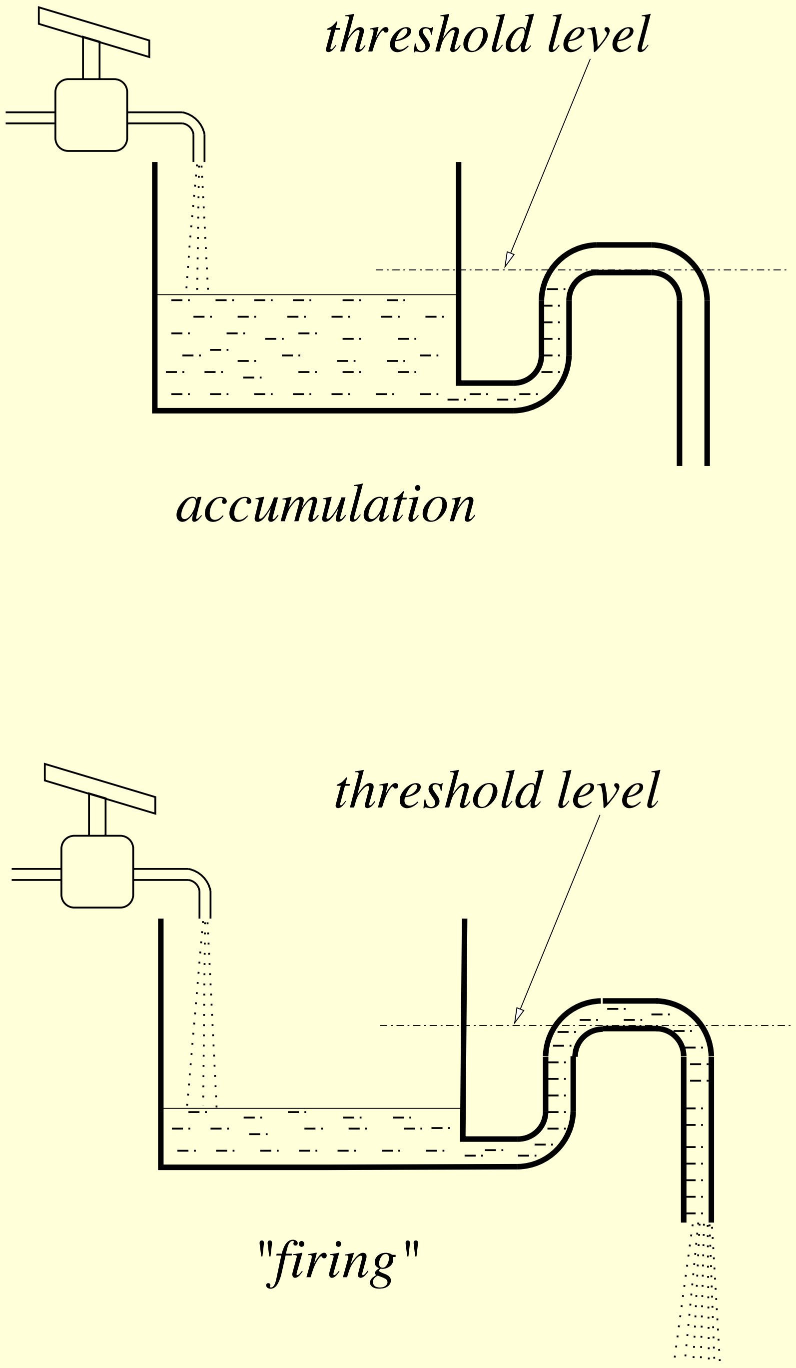
$$\mathbf{x}_0(t) = \mathbf{x}_0(t+T), \; T = 2\pi/\omega$$

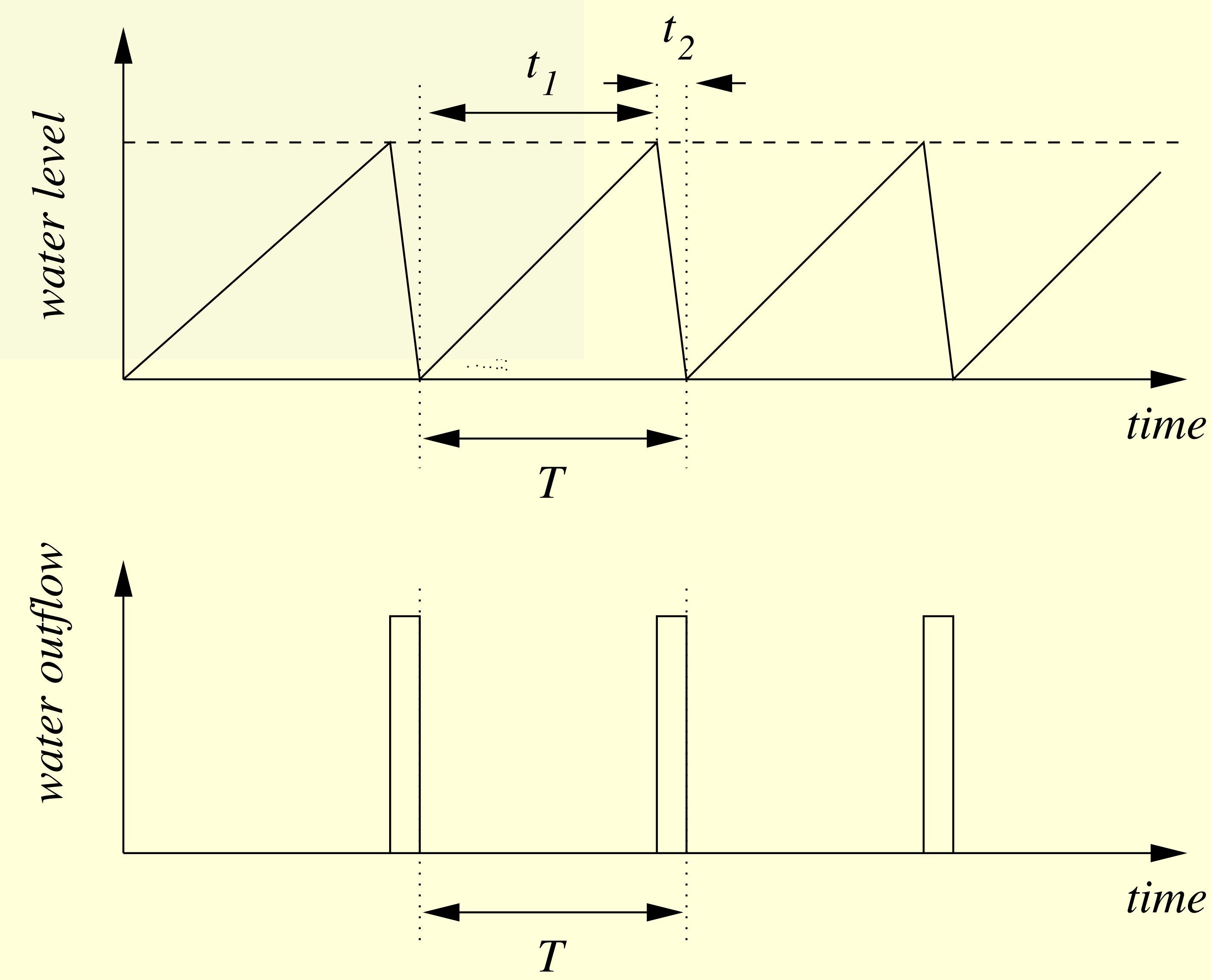
In the **phase space** (the space of all variables $\mathbf{x}$) this solution is represented by an isolated closed attractive curve, called

limit cycle



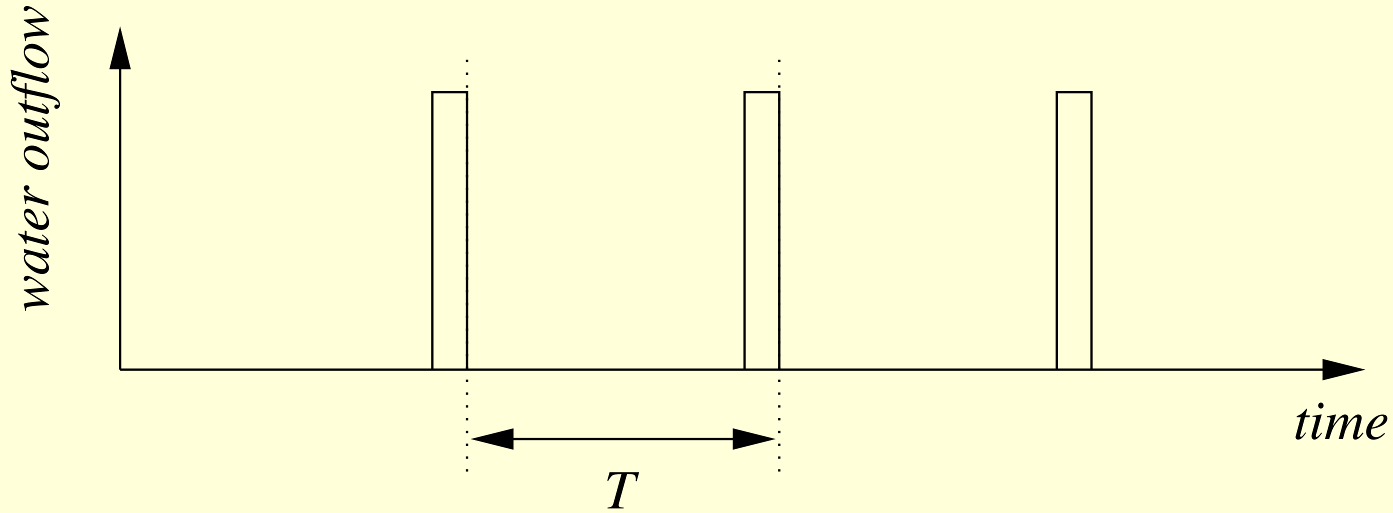
Self-sustained oscillators: example II





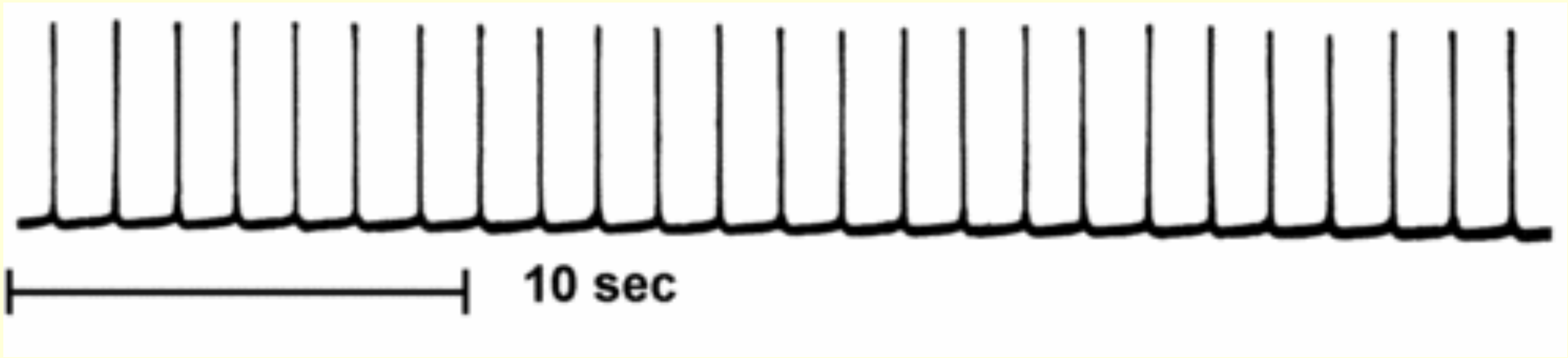
Integrate-and-fire system

Self-sustained oscillators: example II

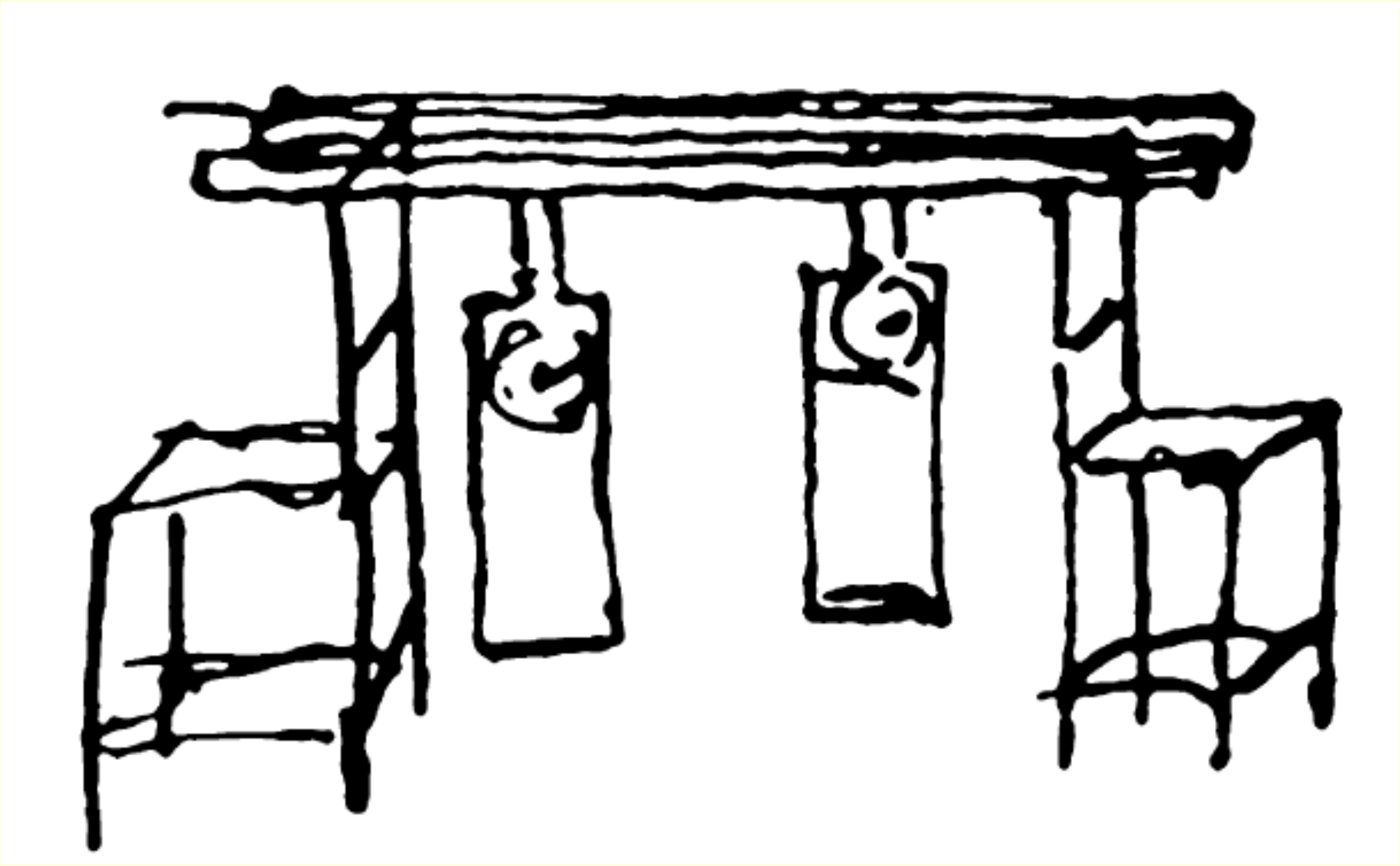


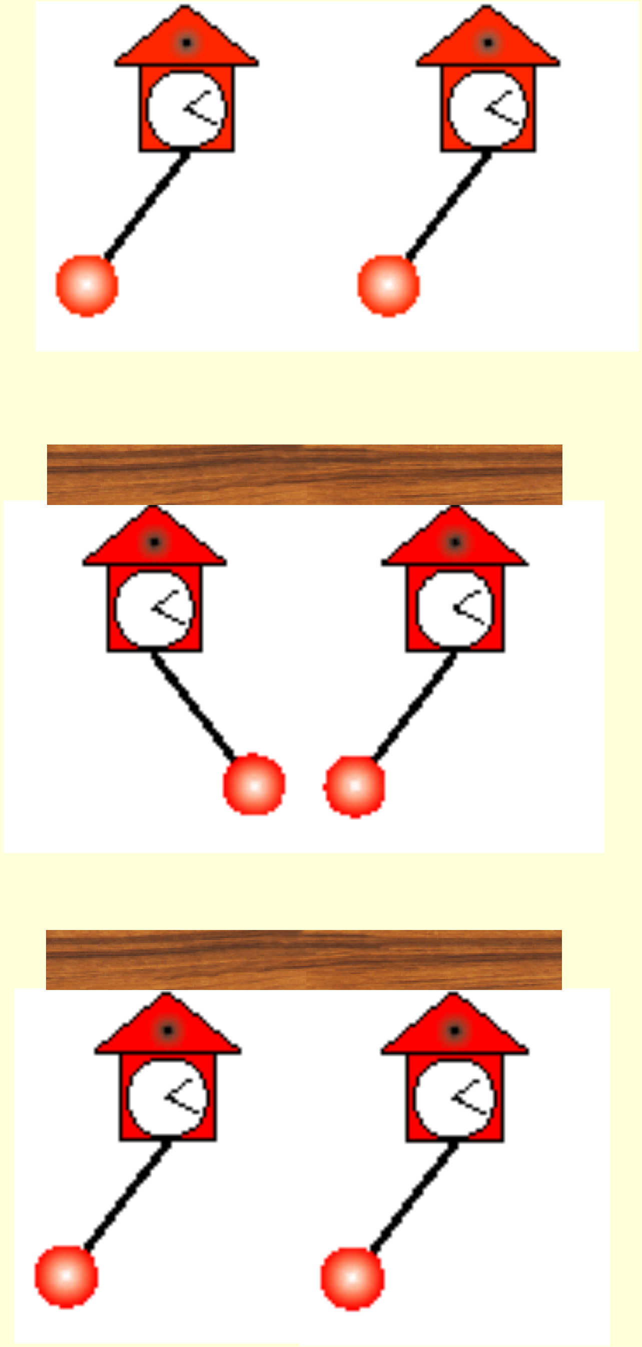
Integrate-and-fire system

is simple but widely used model of **neuron** firing



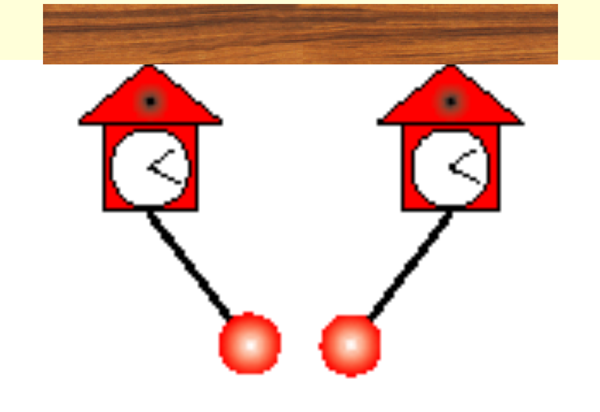
Discovery of synchronization

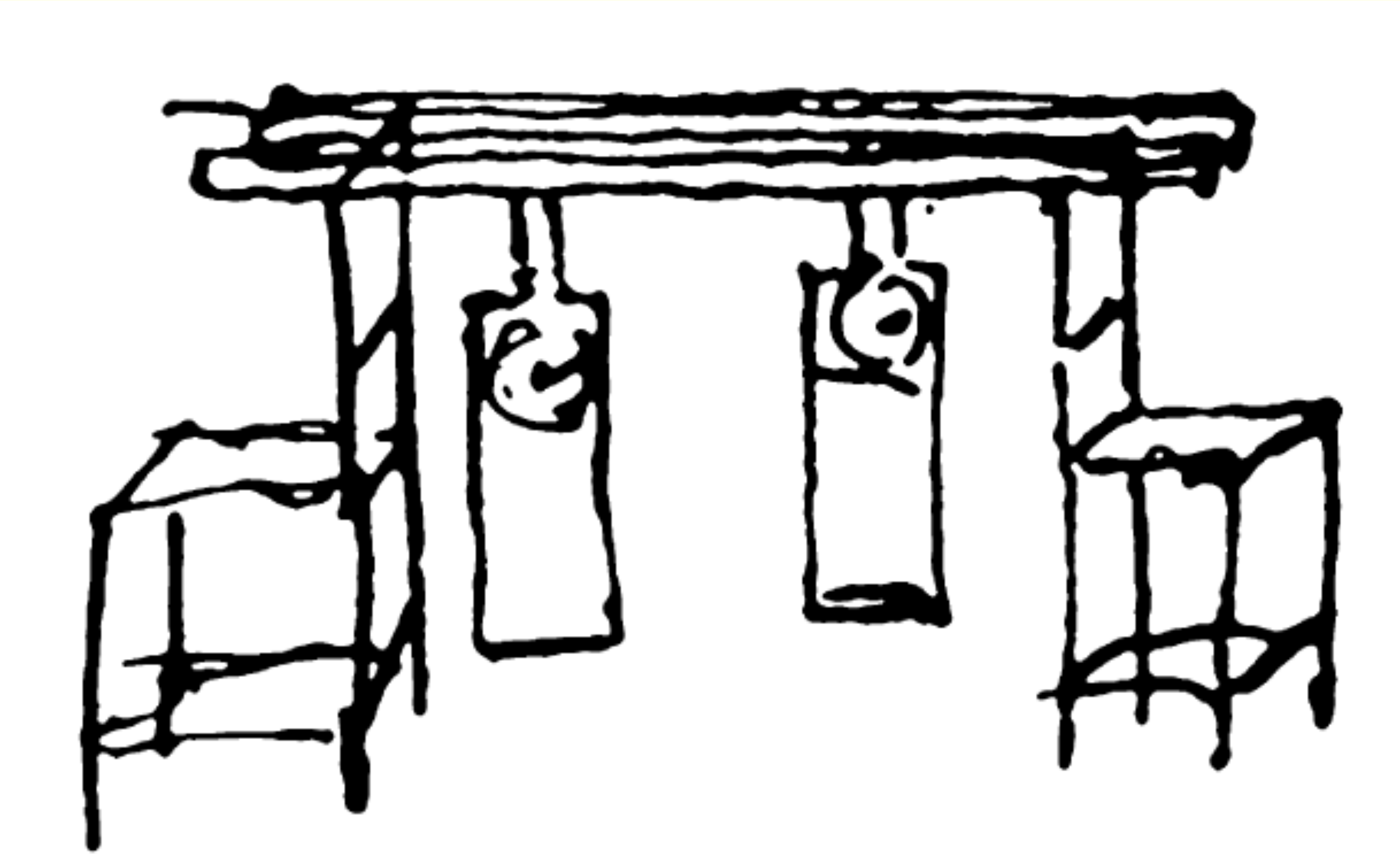




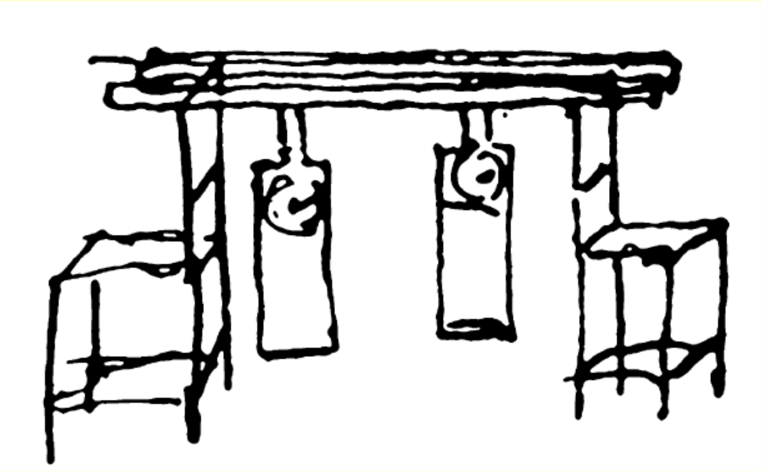
Christiaan Huygens, 1656

Christiaan Huygens: mutual sympathy of clocks





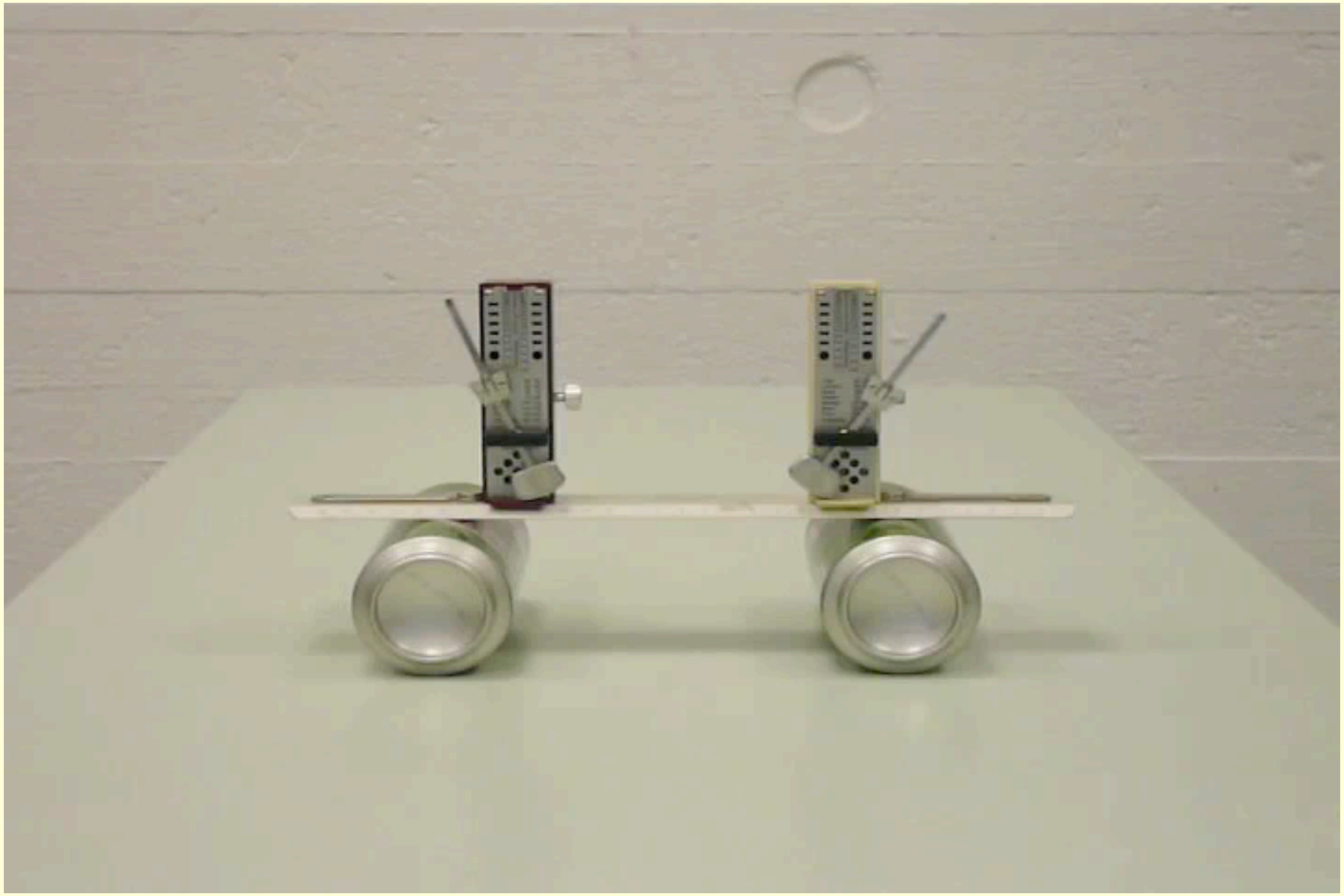
Christiaan Huygens: mutual sympathy of clock II



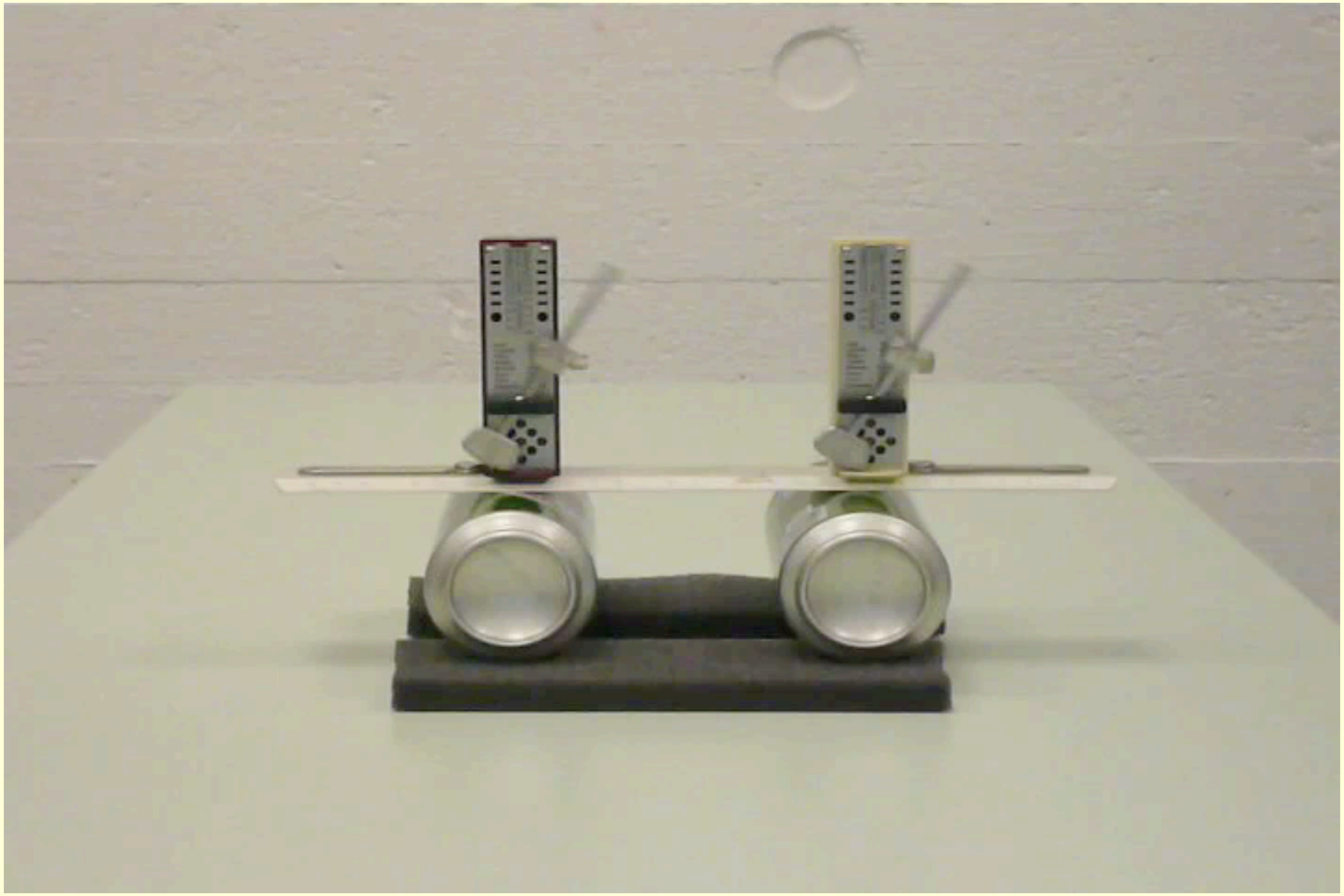
...It is quite worths noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible.

Christiaan Huygens, Horologium Oscillatorium, 1656

Demonstration of synchronization I



Demonstration of synchronization II

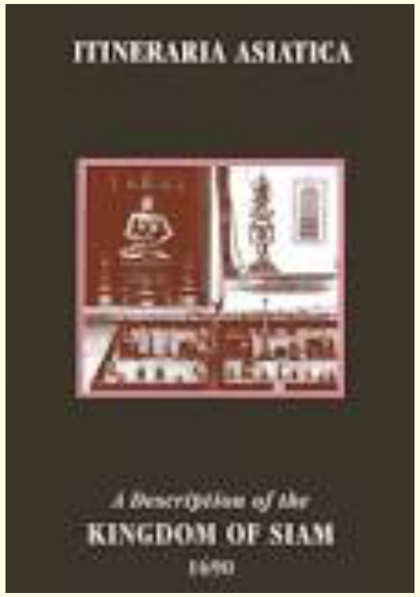


Many metronomes on a moveable support

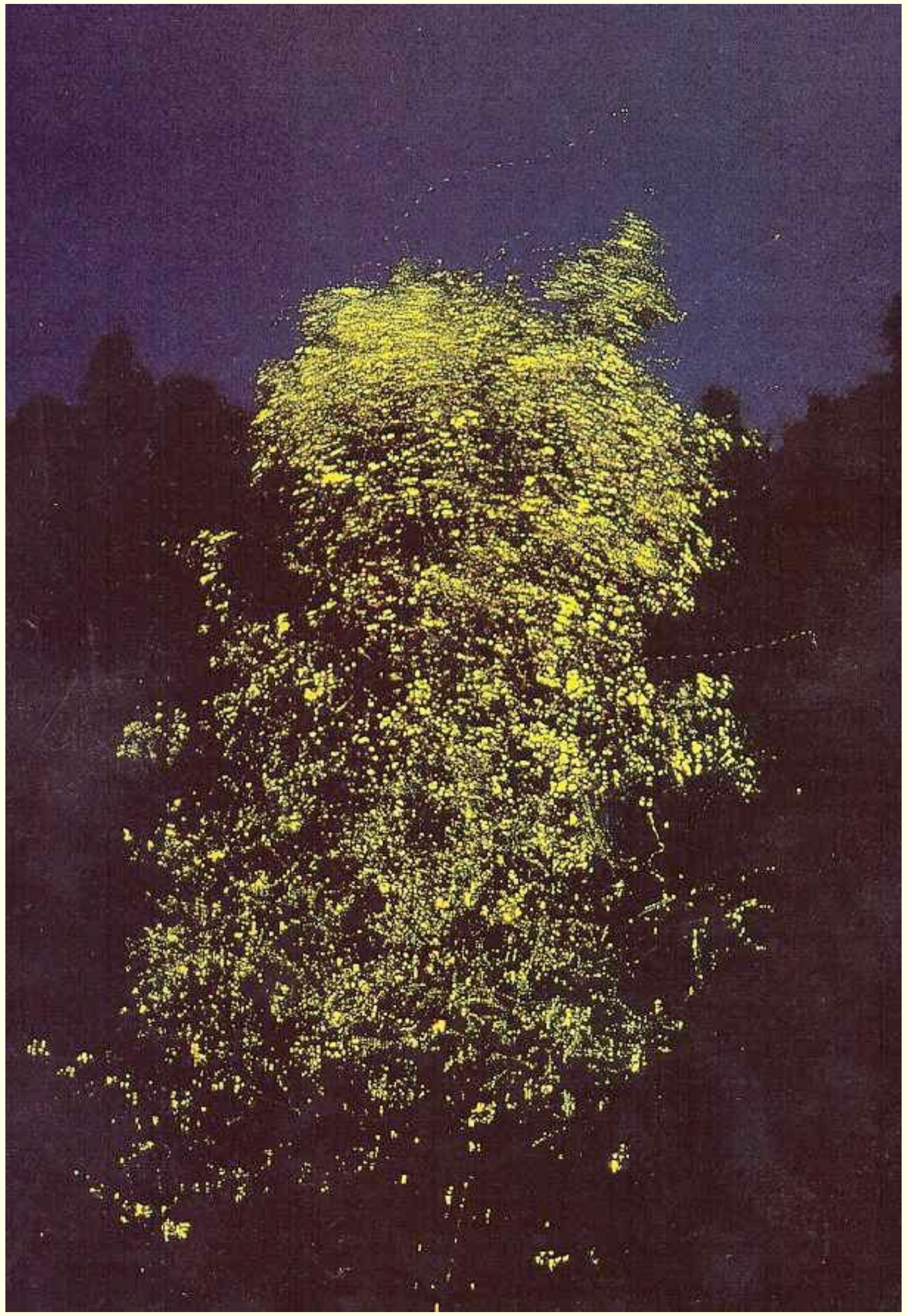
Fireflies synchrony

Engelbert Kaempfer

(16.09.1651, Lemgo, Germany - 2.11.1716)

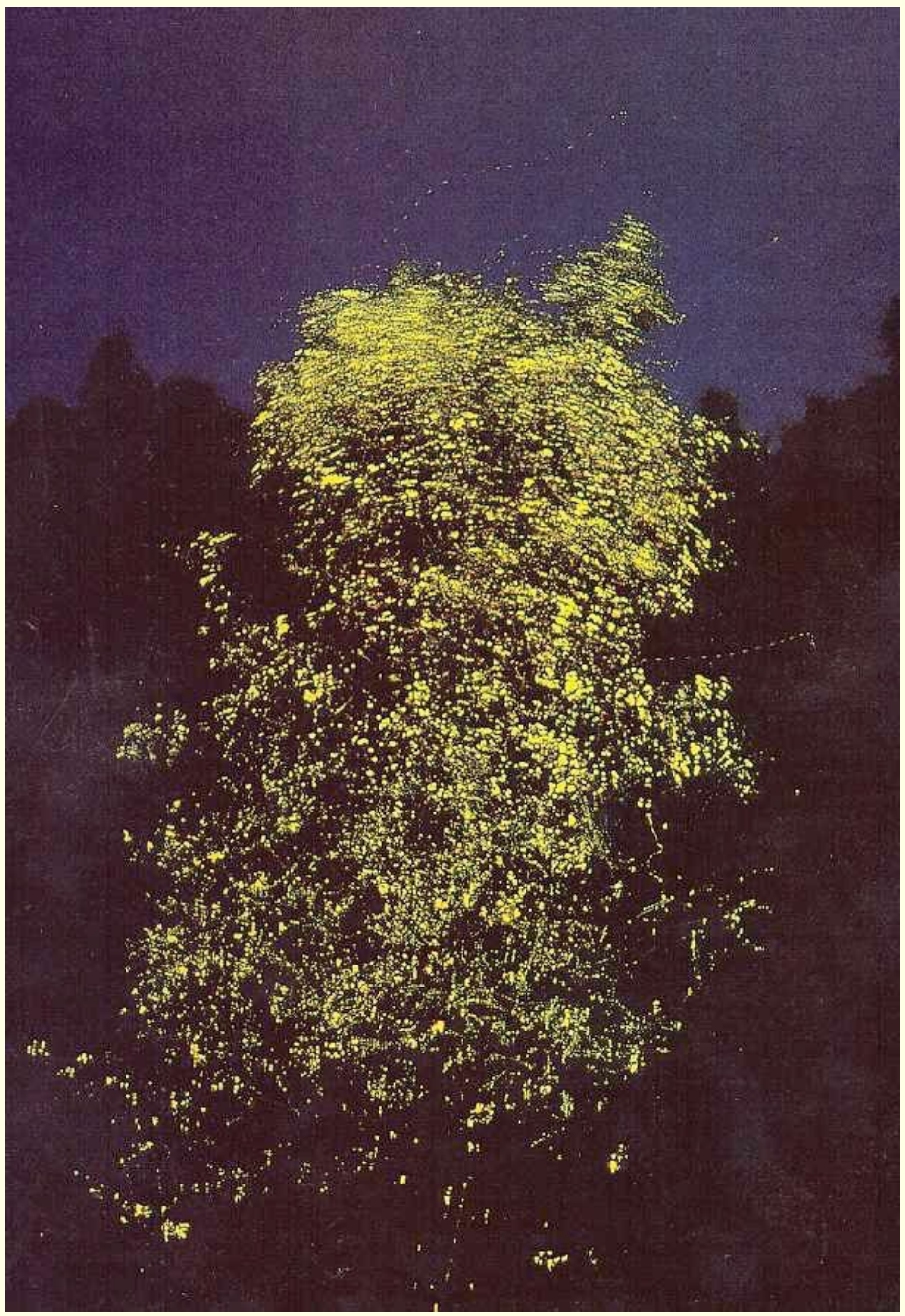


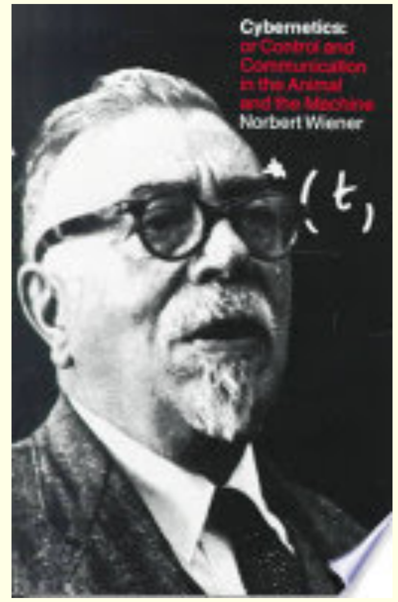
A description of the Kingdom of Siam, 1690



Fireflies "hide their Lights all at once, and a moment after make it appear again with the utmost regularity and exactness."

Fireflies synchrony II



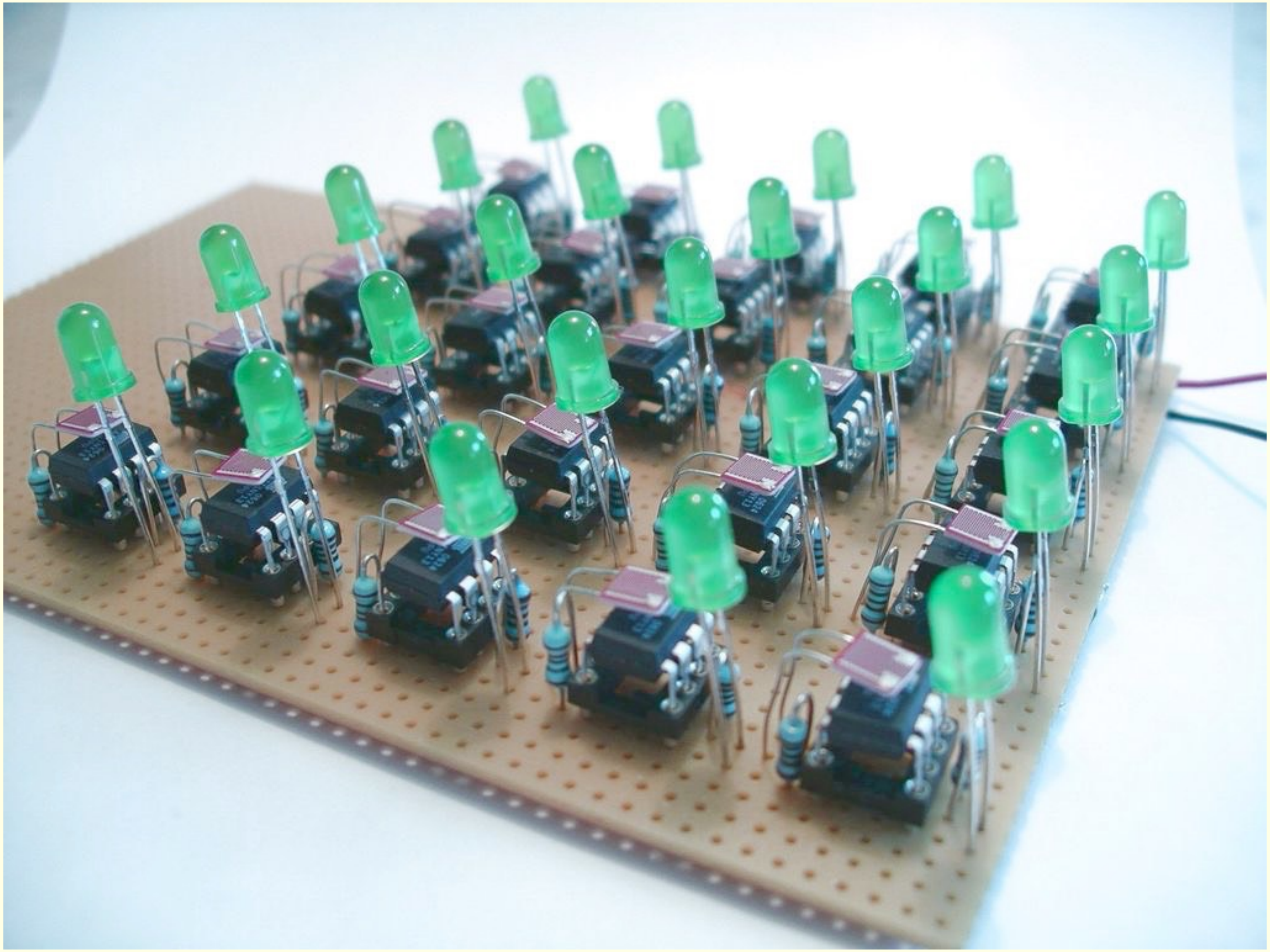


Norbert Wiener

Cybernetics: or the Control and Communication in the Animal and the Machine, 1961

Hypothesis: same "*phenomenon of the pulling together of frequencies*" is responsible for emergence of the brain waves

Electronic "Fireflies"



“Bikeflies”

"Bikeflies"

1,000
Fireflies

1050 W Wilson, Chicago, IL 60640

Share:

Buy your synchronizing bike light here and participate! For the September 27 Chicago premiere of The Kuramoto Model (1,000 Fireflies), 250 custom bike lights will be distributed to cyclists attending the EdgeUp festival, part of Chicago Artists Month. Using radio communication, these devices synchronize their blinking patterns with other nearby devices, altering social rules of proximity and generating a nomadic self-organizing system.

https://squareup.com/market/1-000-fireflies

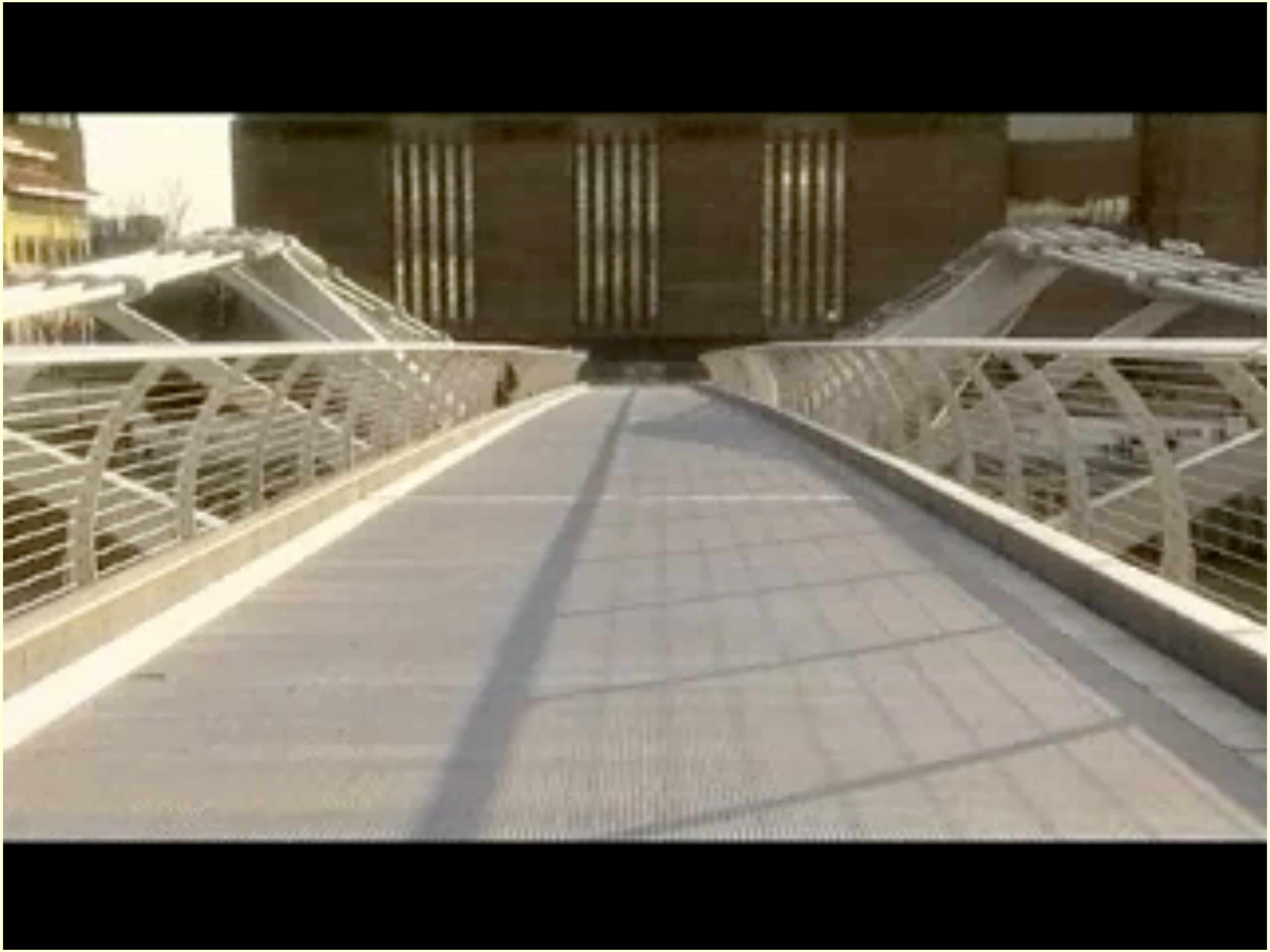
Pedestrian synchrony on footbridges

Millennium Bridge, London, river Thames

Millennium Bridge, opening day

Film: Arup Group Limited

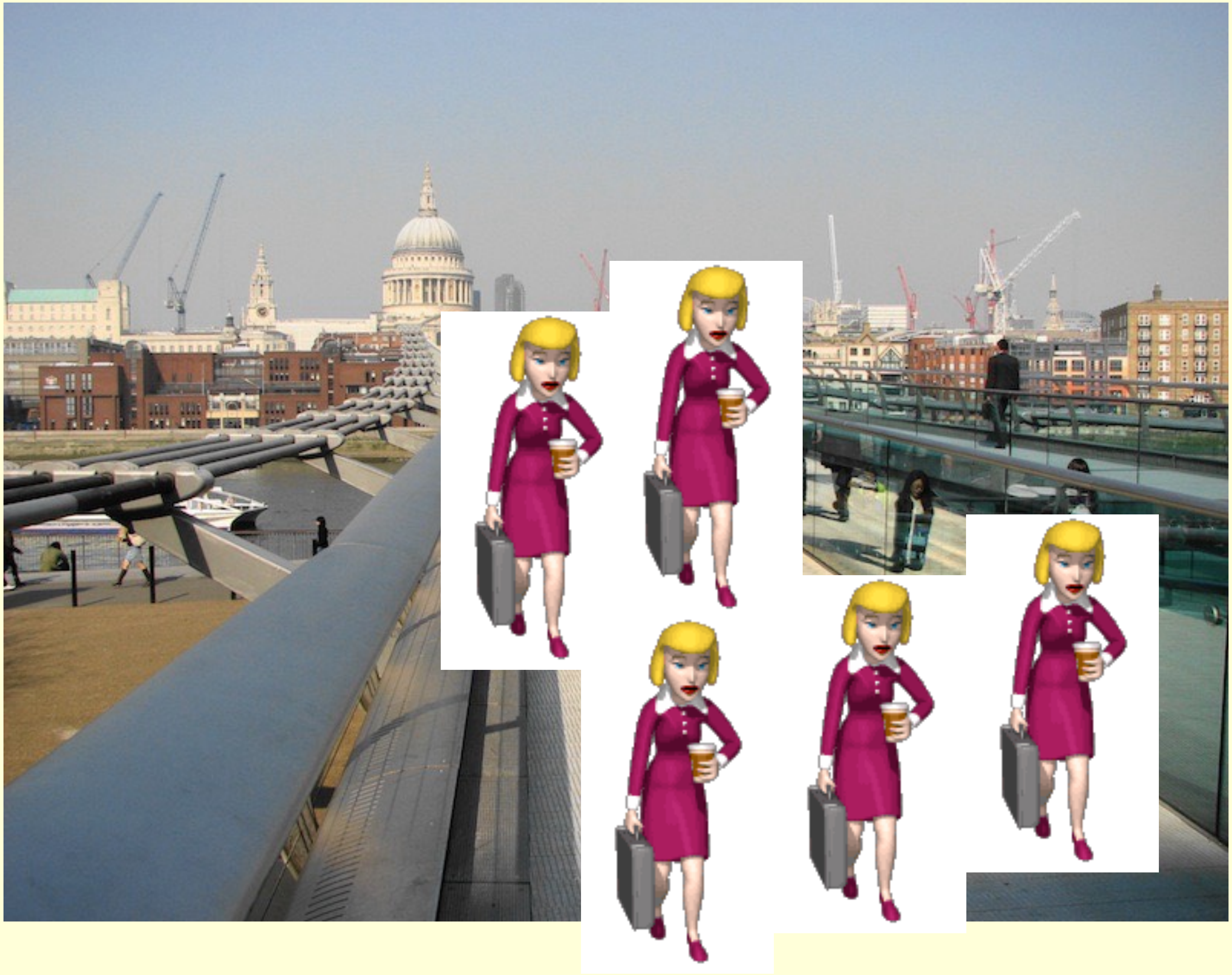
Synchronization vs. resonance



Film: Arup Group Limited

Synchronization vs. resonance II

Resonance: The force originally exists, the system (the bridge) responds to it



Synchronization vs. resonance III

Synchronization: Originally there is no force, but it emerges due to self-organization

Synchronized applause

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brief communications

Z. Néda*, E. Ravasz*, Y. Brechet†,T. Vicsek‡, A.-L. Barabási

The sound of many hands clapping

Tumultuous applause can transform itself into waves of synchronized clapping.

An audience expresses appreciation for a good performance by the strength and nature of its applause. The thunder of applause at the start often turns quite suddenly into synchronized clapping, and this synchronization can disappear and reappear several times during the applause. The phenomenon is a delightful expression of social self-organization that provides an example on a human scale of the synchronization processes that occur in numerous natural systems, ranging from flashing Asian fireflies to oscillating chemical reactions[1–3].

… in the smaller and culturally more homogeneous eastern European communities, synchronized clapping is a daily event, whereas it happens only sporadically in western European and North American audiences.

Synchronization: main problems

1 Externally forced oscillator

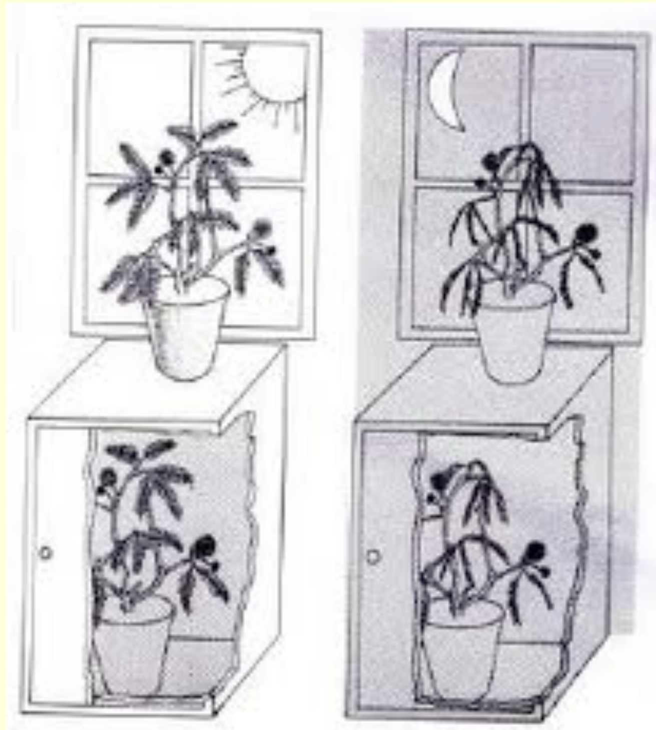
2 Two mutually coupled oscillators

3 Several coupled oscillators

4 Large population of oscillators

5 Chaotic systems

Entrainment by an external force: circadian rhythms

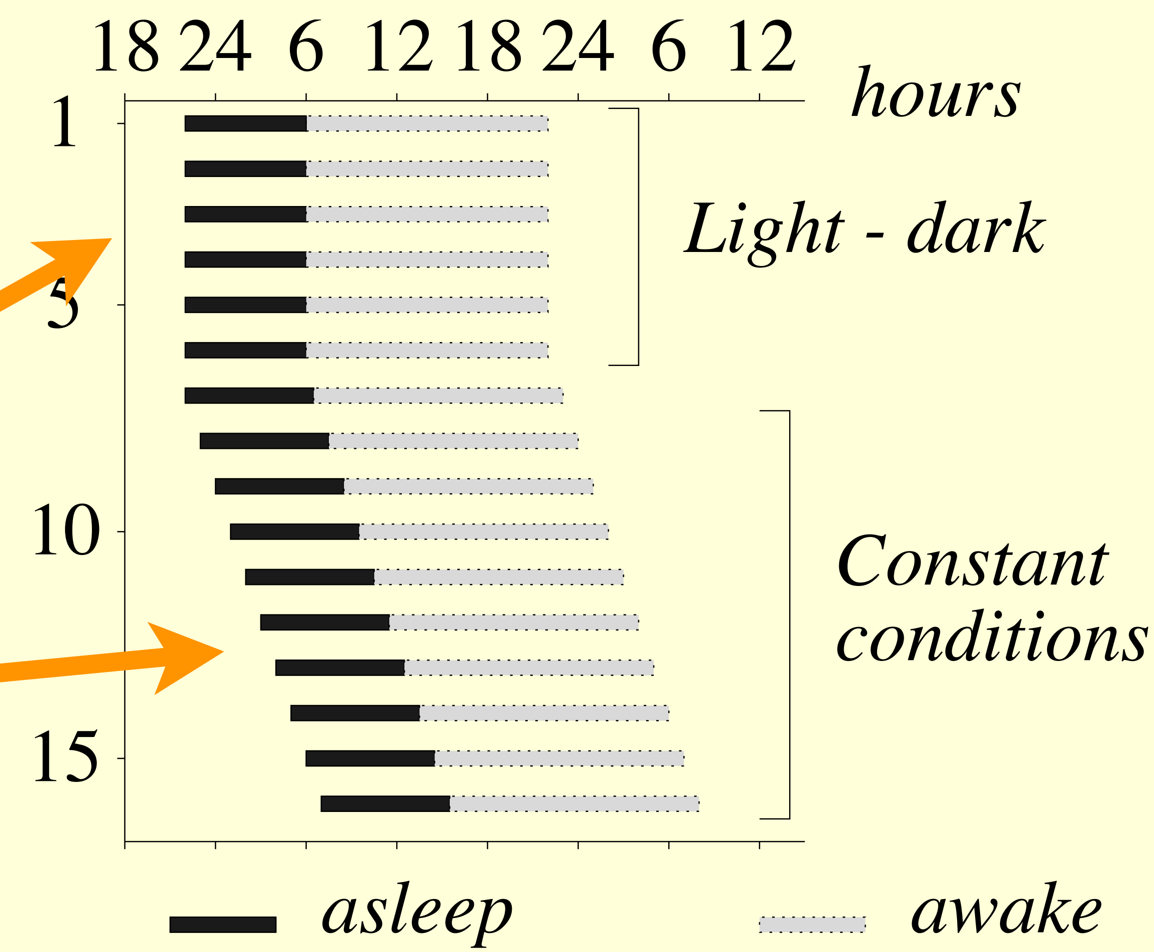


Motion of leaves of a plant in the darkness: evidence for the existence of the inner clock (circadian rhythm)

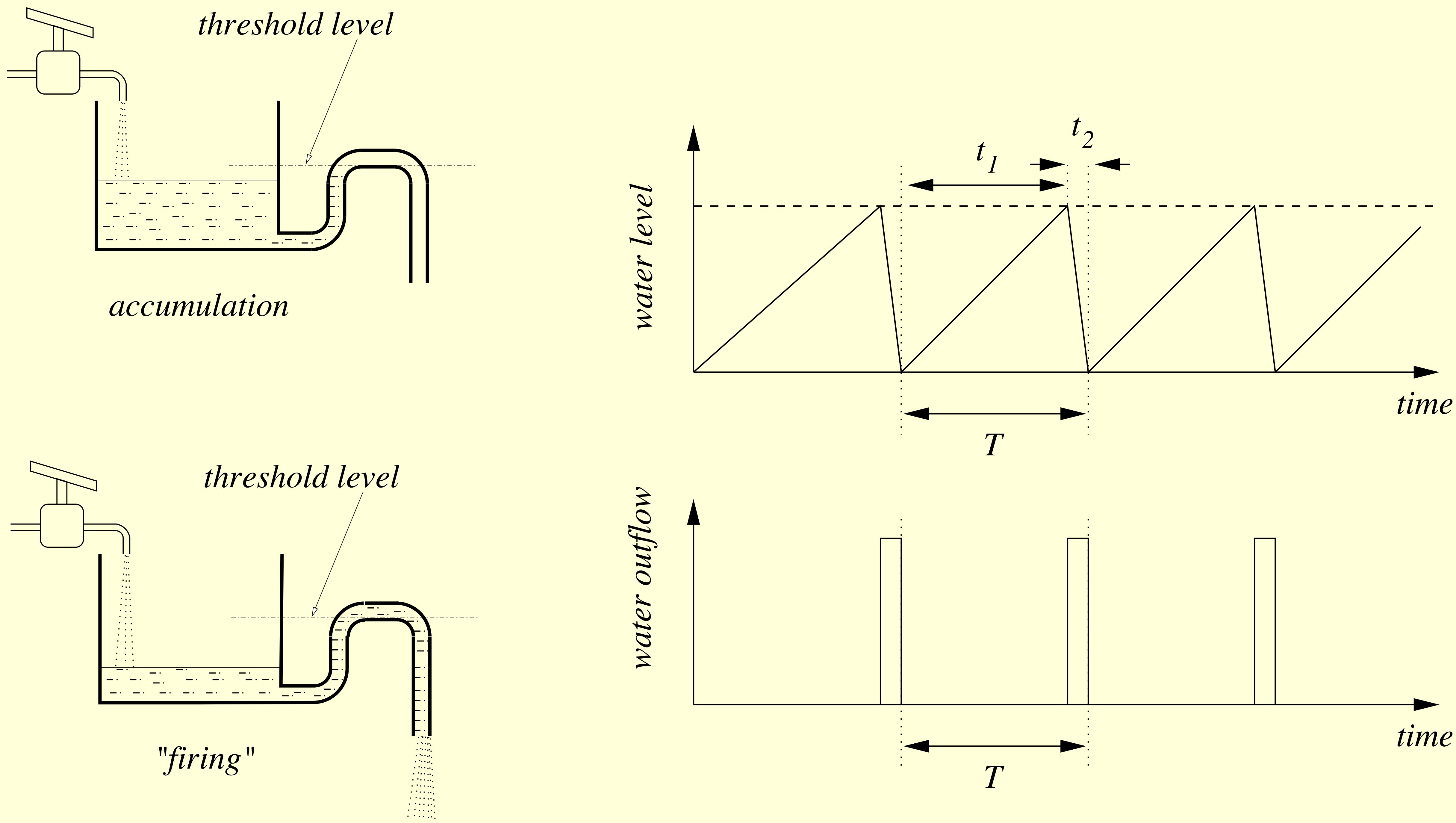
Jean-Jacques d'Ortous de Mairan, 1729

Inner clock is locked to the light-dark cycle

Loss of synchrony

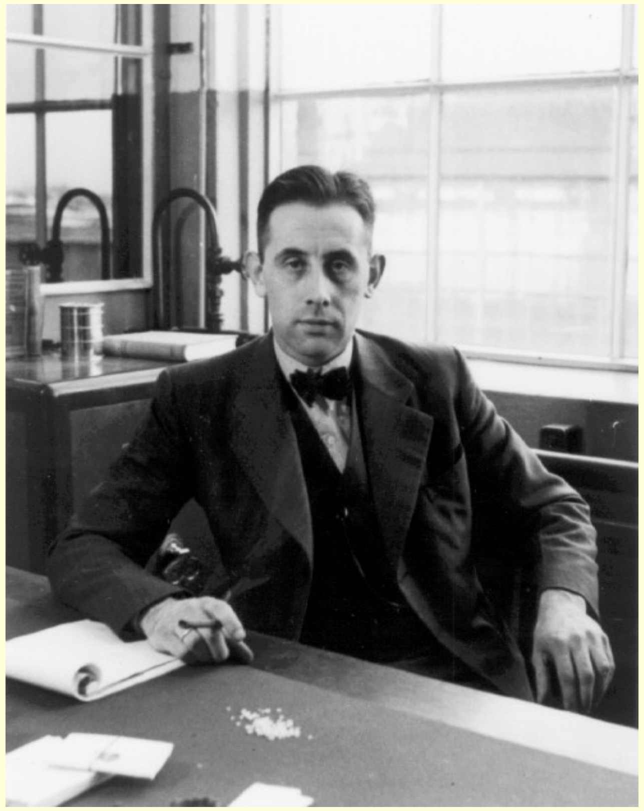


Self-sustained oscillators: example II

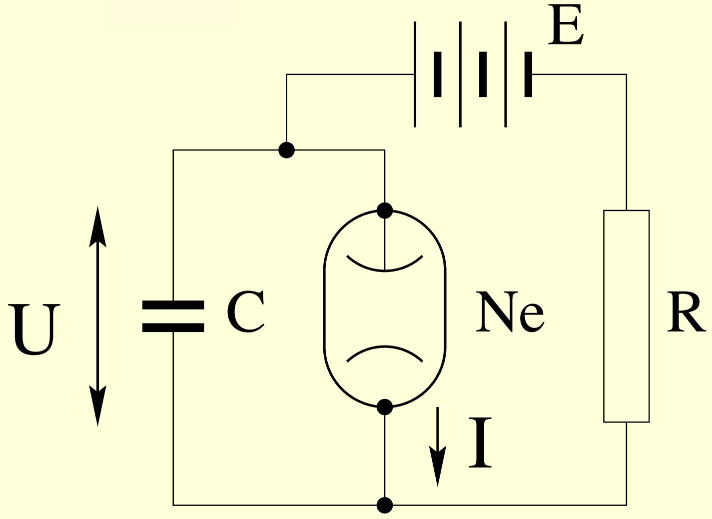


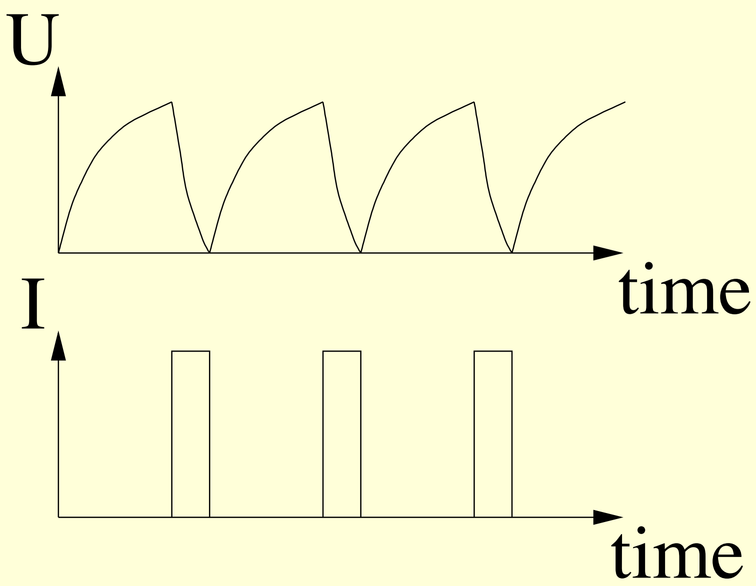
Integrate-and-fire system

Entrainment by an external force: neon tube oscillator



Van der Pol, 1926





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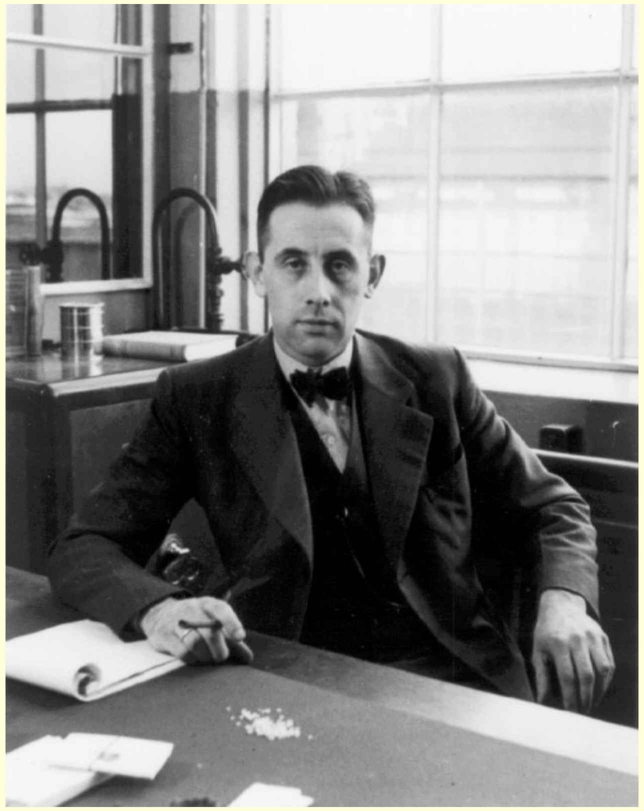

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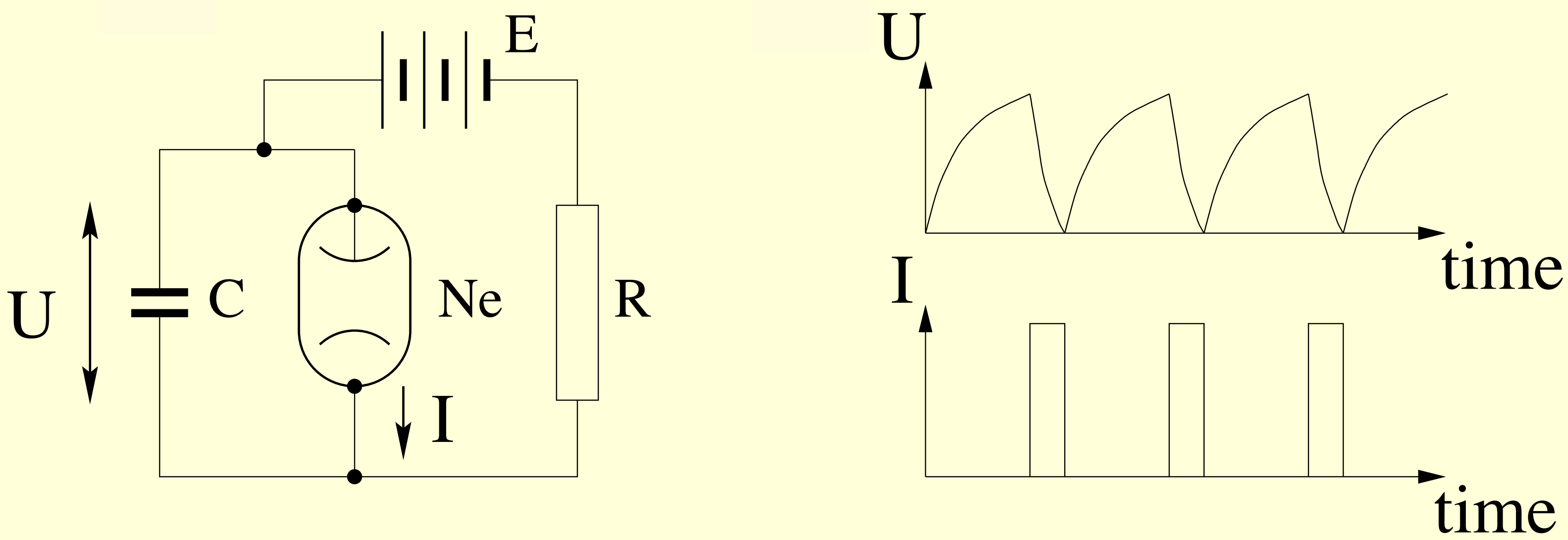

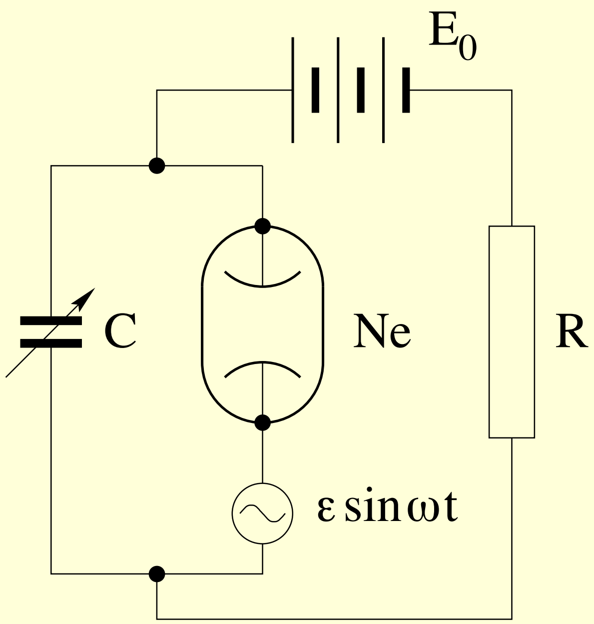

forced system

Van der Pol, van der Mark, 1927

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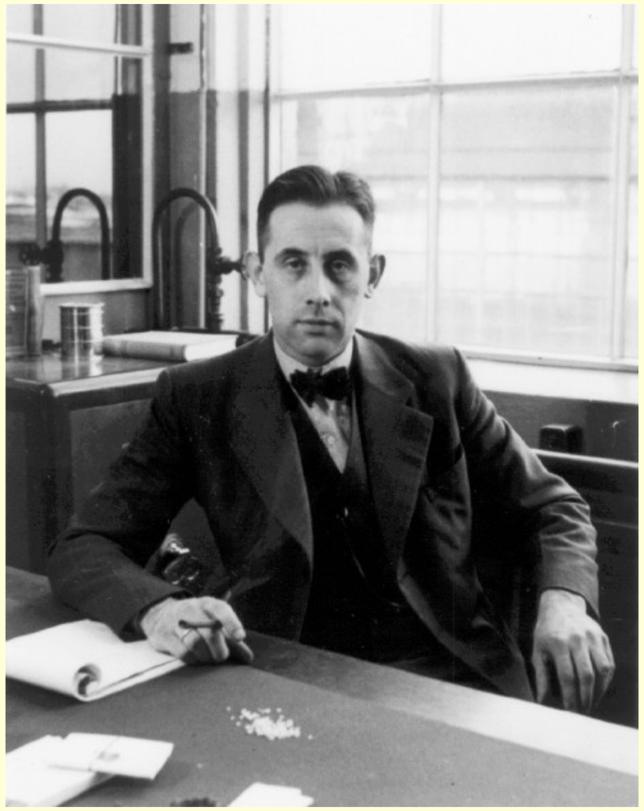

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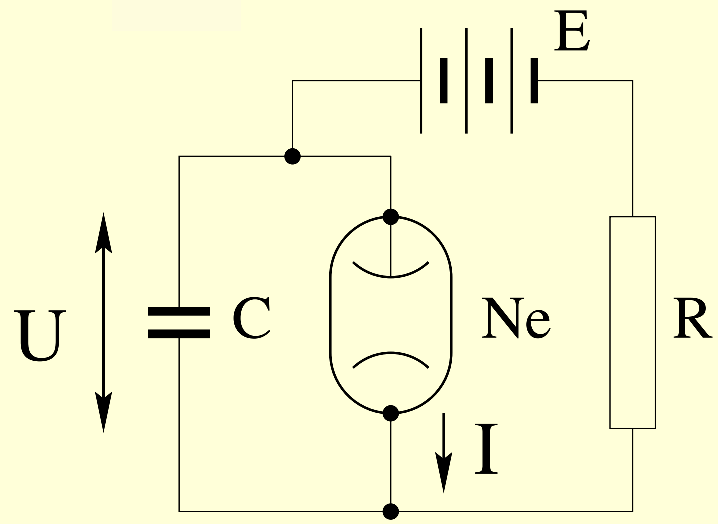

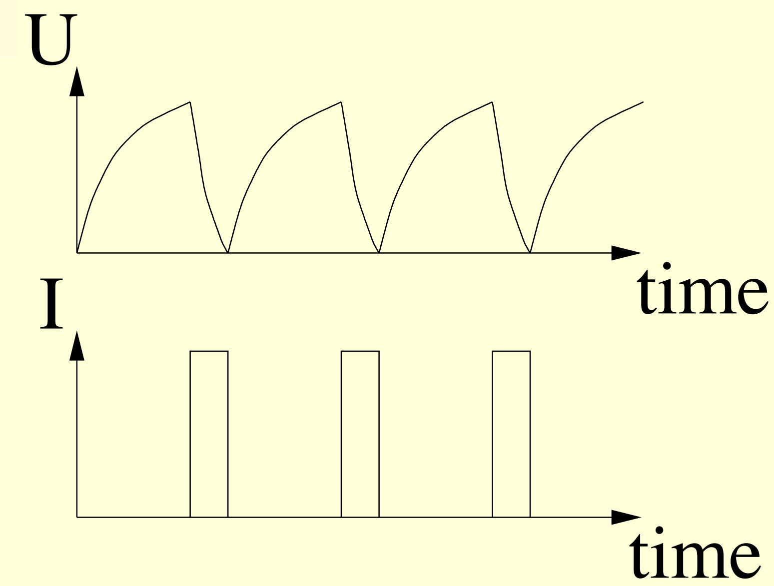

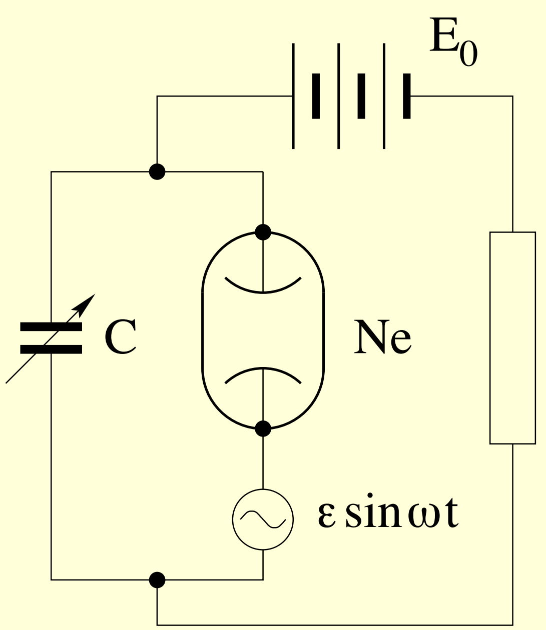

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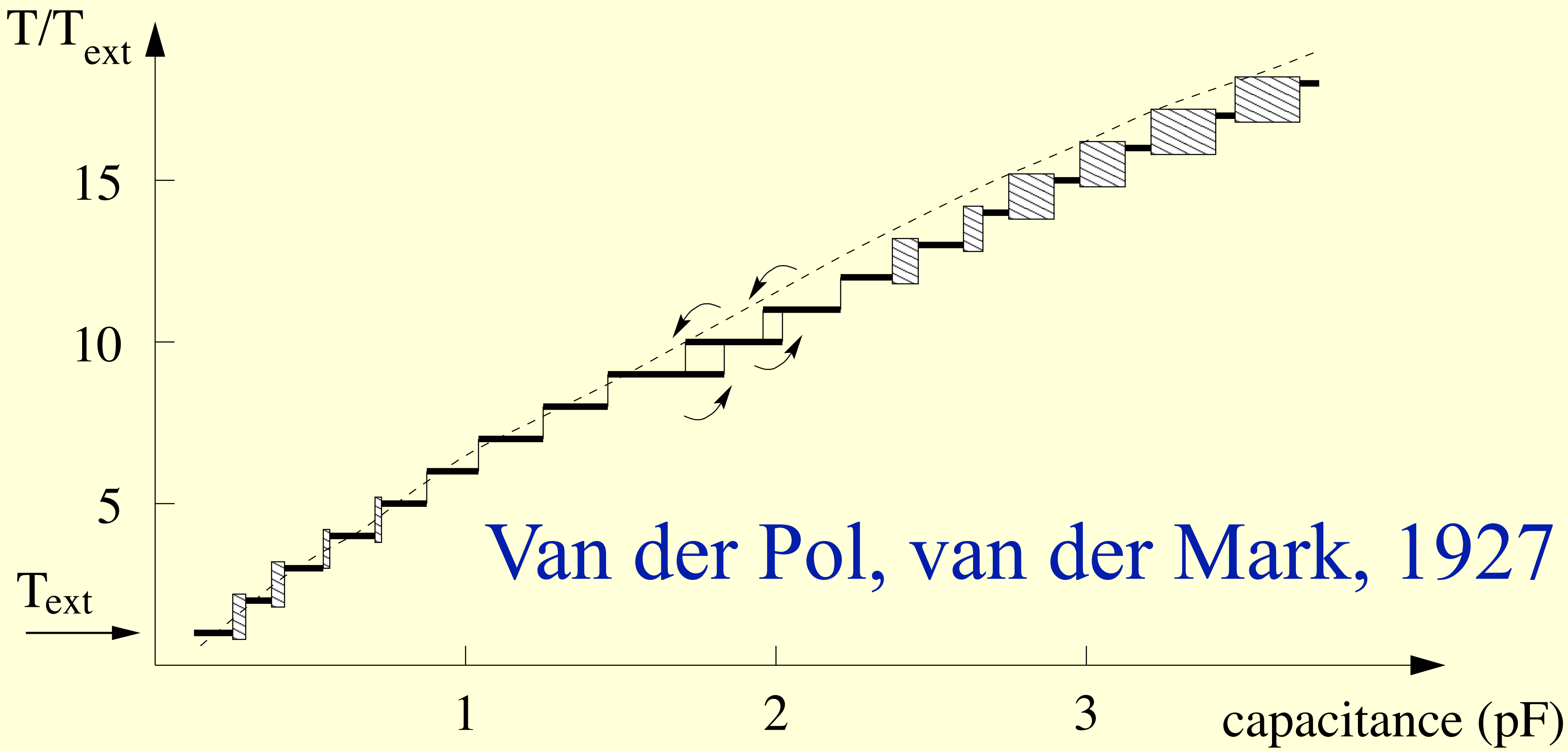

Van der Pol, van der Mark, 1927

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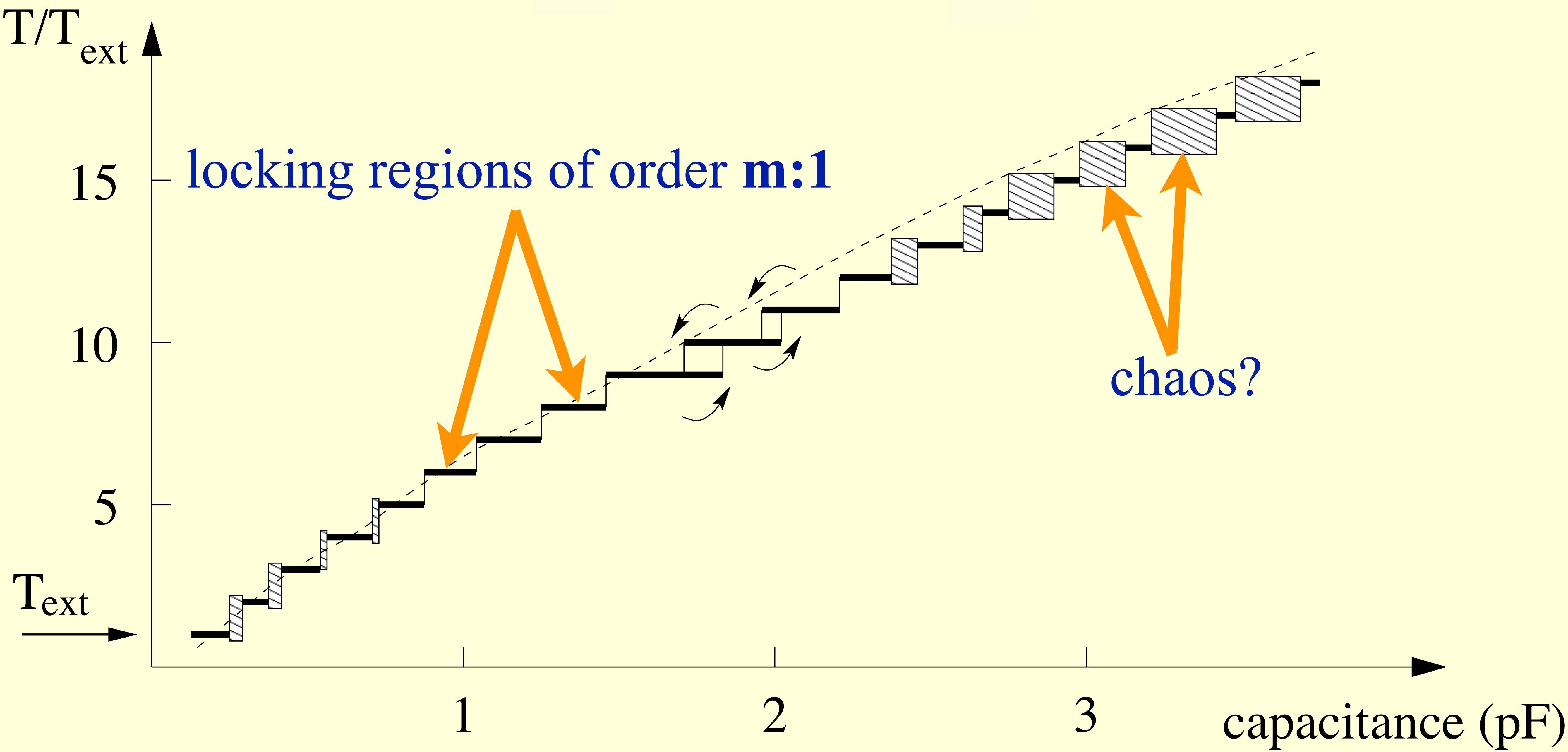
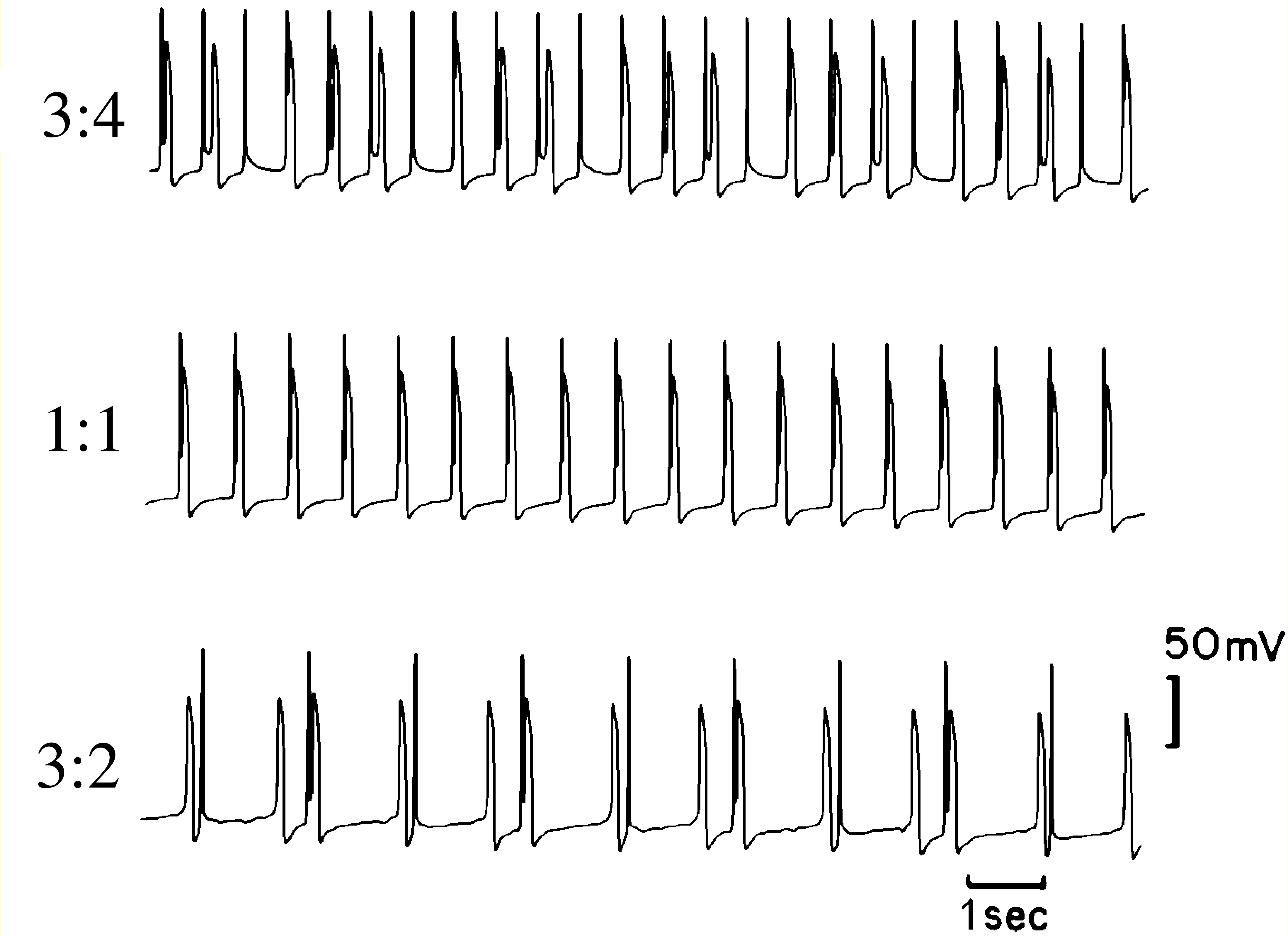


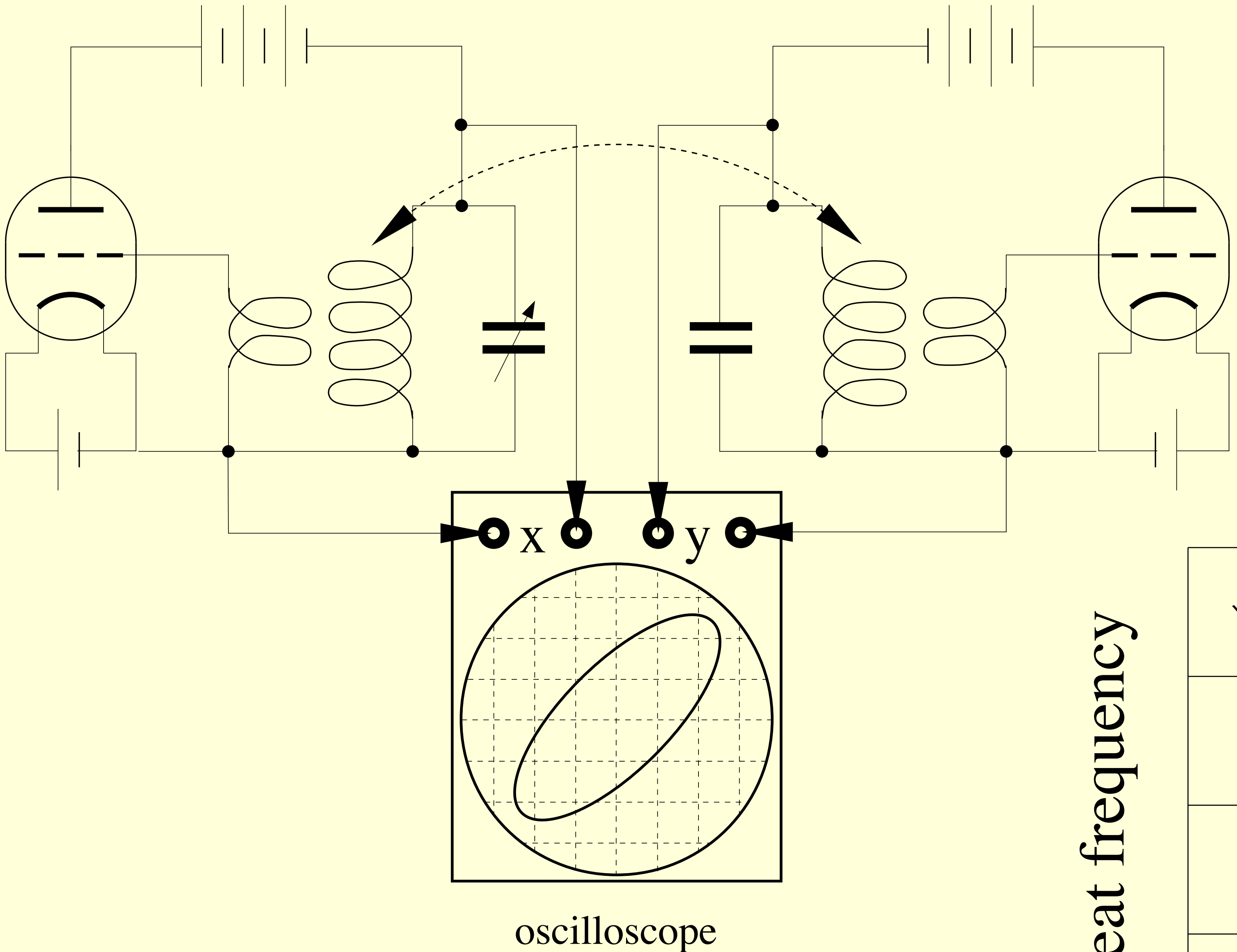
Entrainment by an external force: n:m locking

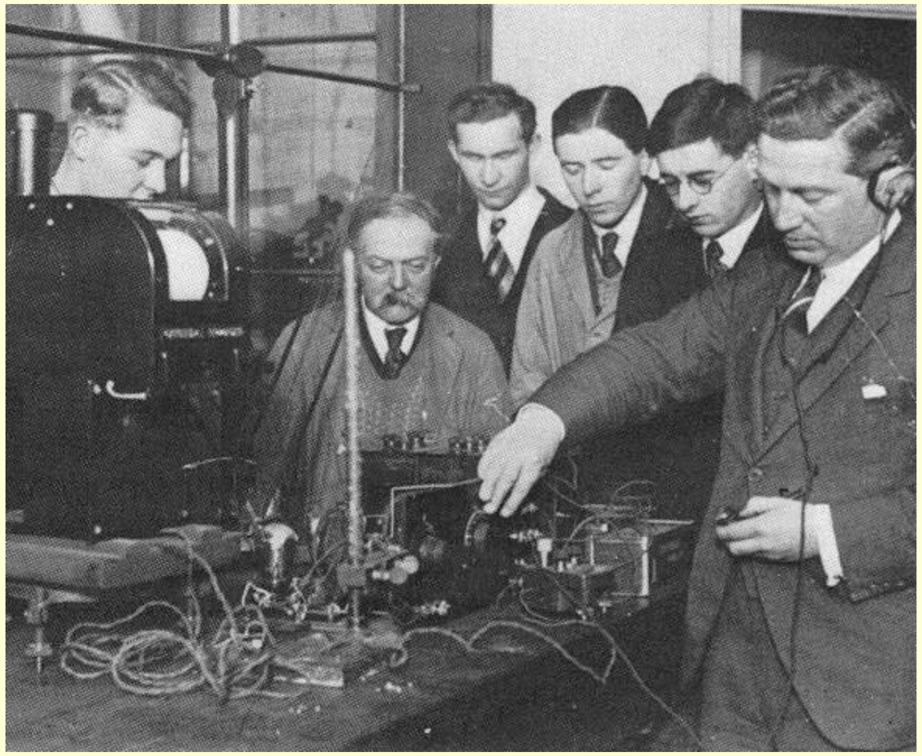


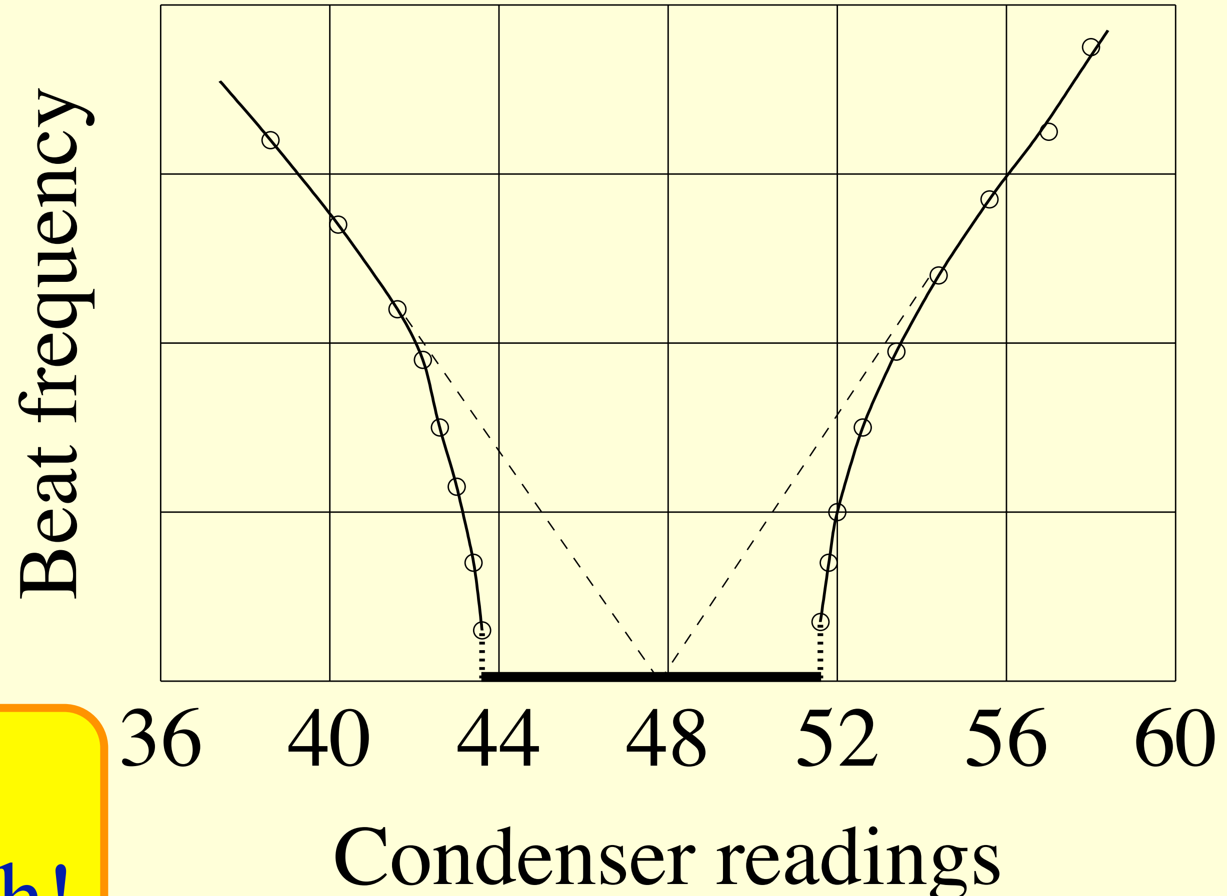
Zeng et al., 1990

Stimulation of embryonic chick atrial heart cells

Two mutually coupled oscillators: classical experiment by Appleton, 1922

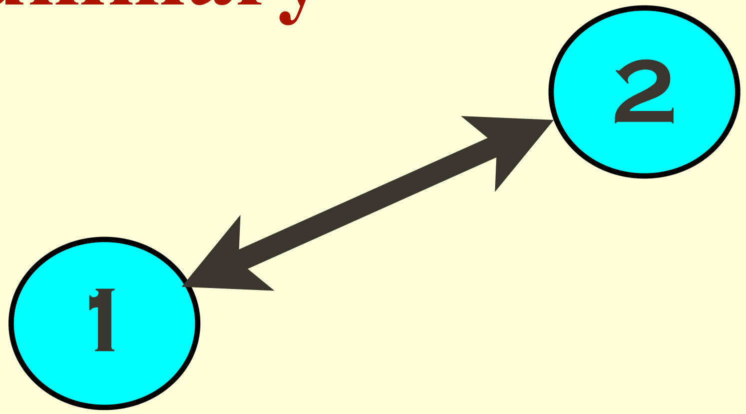




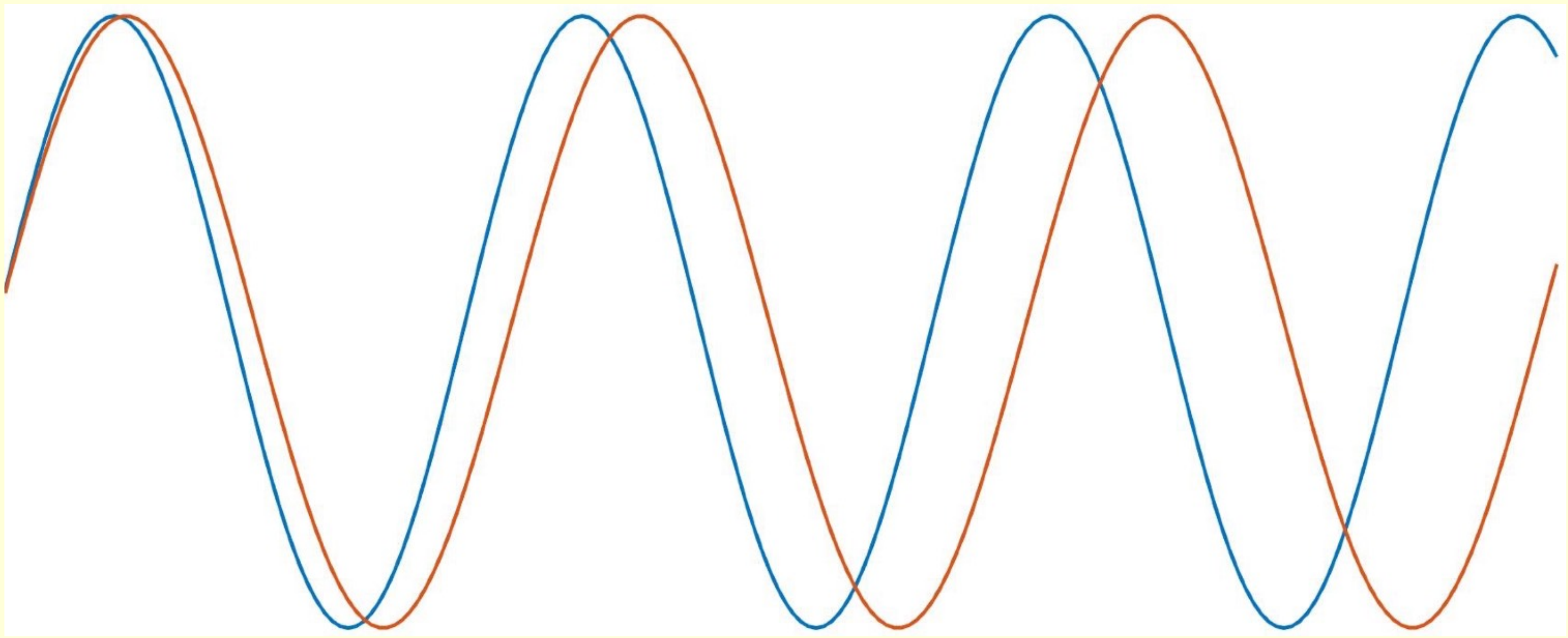


Condition of locking is fulfilled for a finite range of frequencies mismatch!

Synchronization: intermediate summary

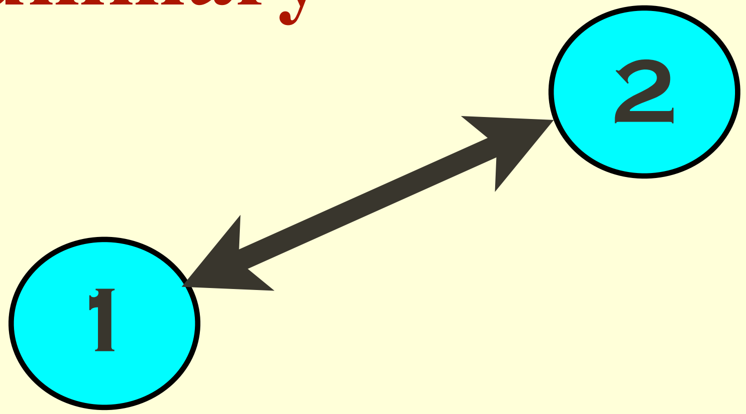


Asynchronous state: <u>different frequencies</u>

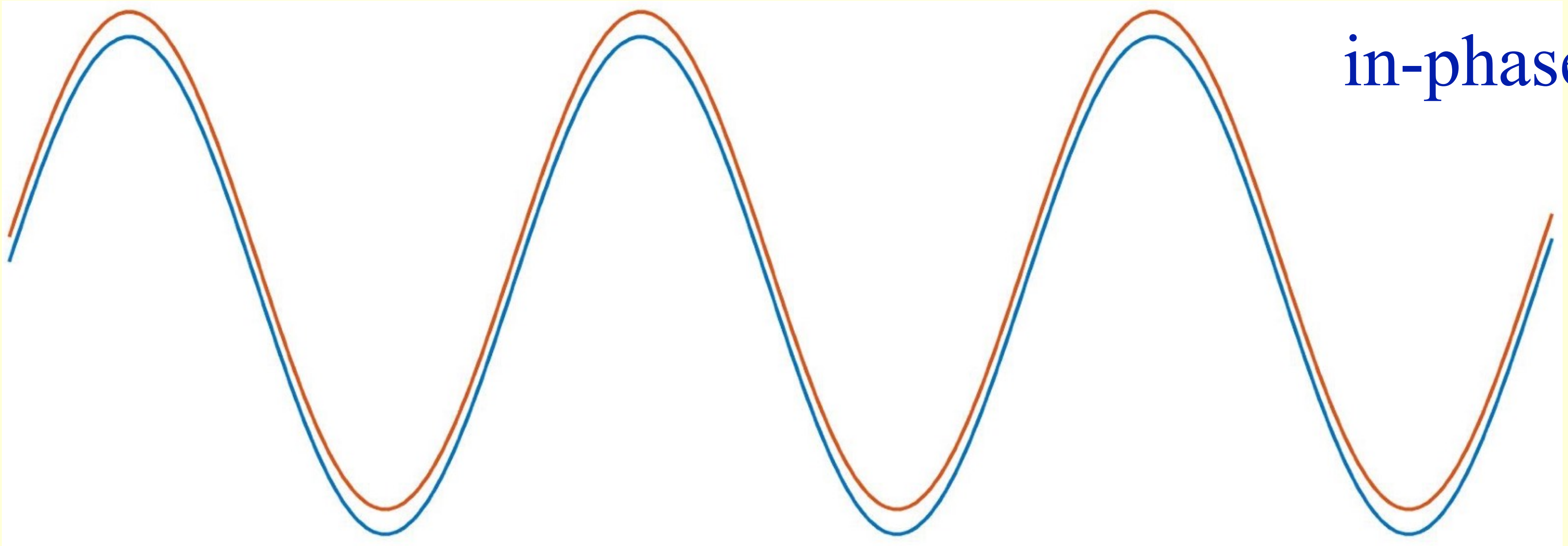


Synchronous state: <u>equal frequencies</u>

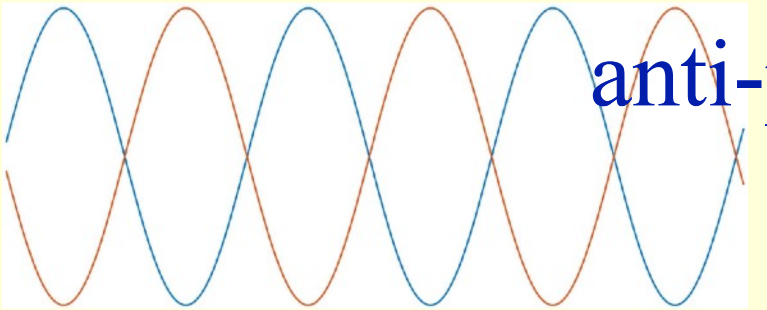
Synchronization: intermediate summary



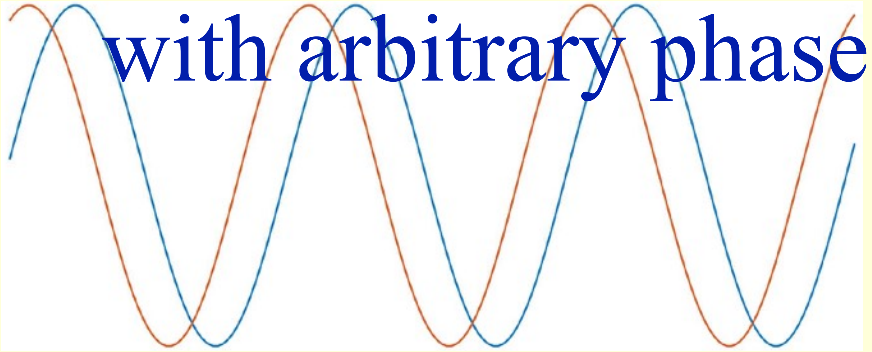
Synchronous state: equal frequencies



in-phase synchrony

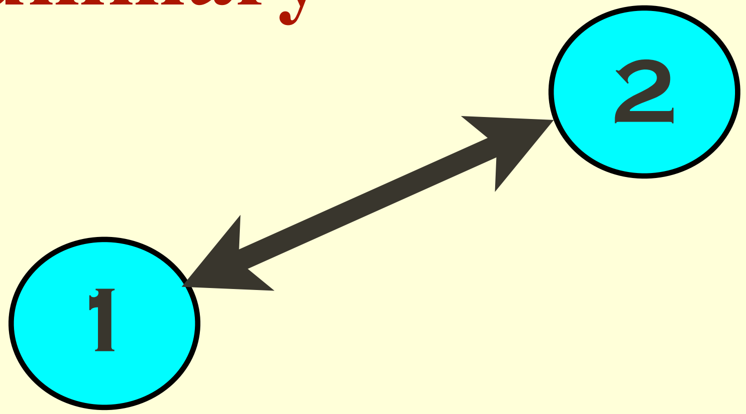


anti-phase synchrony

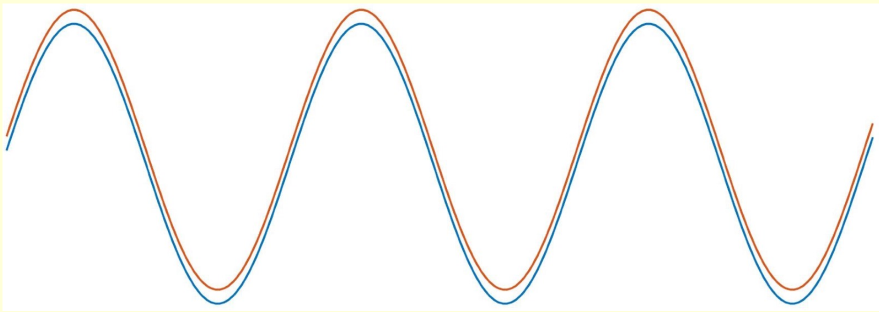


with arbitrary phase shift

Synchronization: intermediate summary

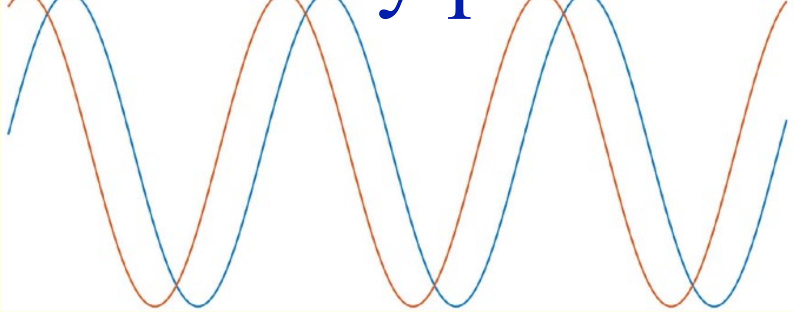


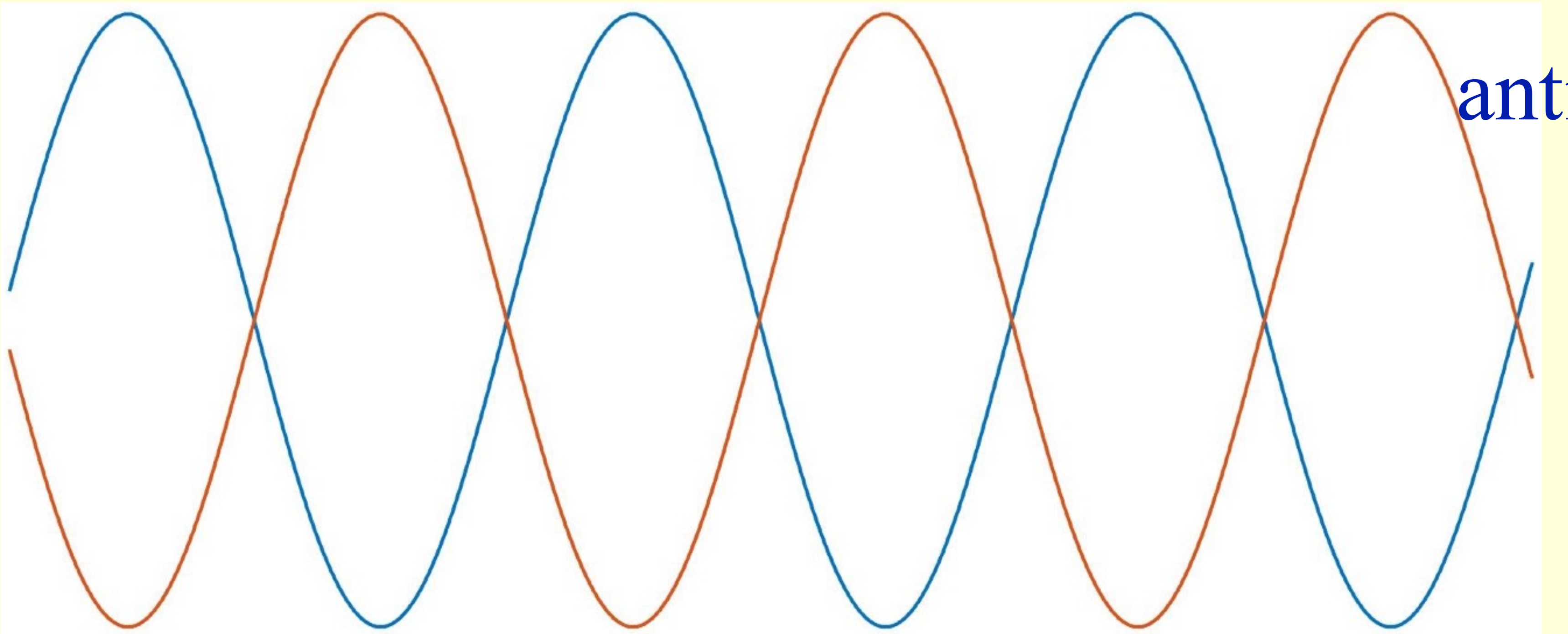
Synchronous state: <u>equal frequencies</u>



in-phase synchrony

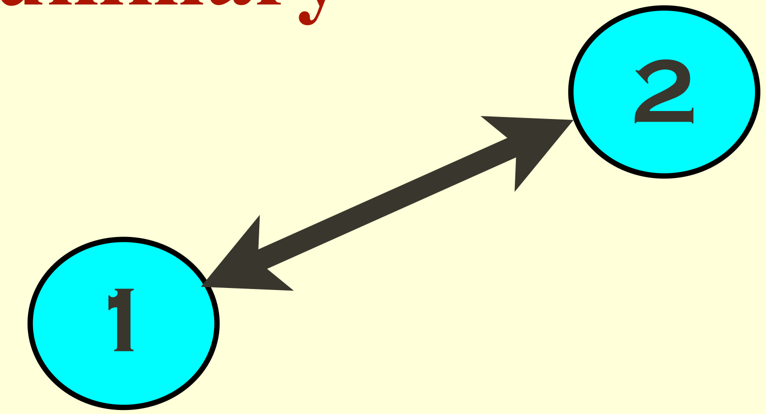
with arbitrary phase shift



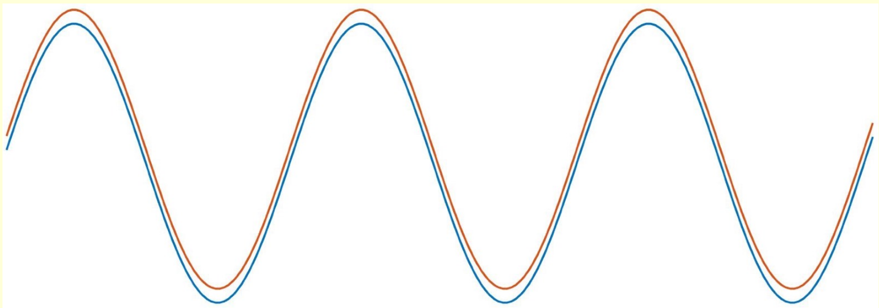


anti-phase synchrony

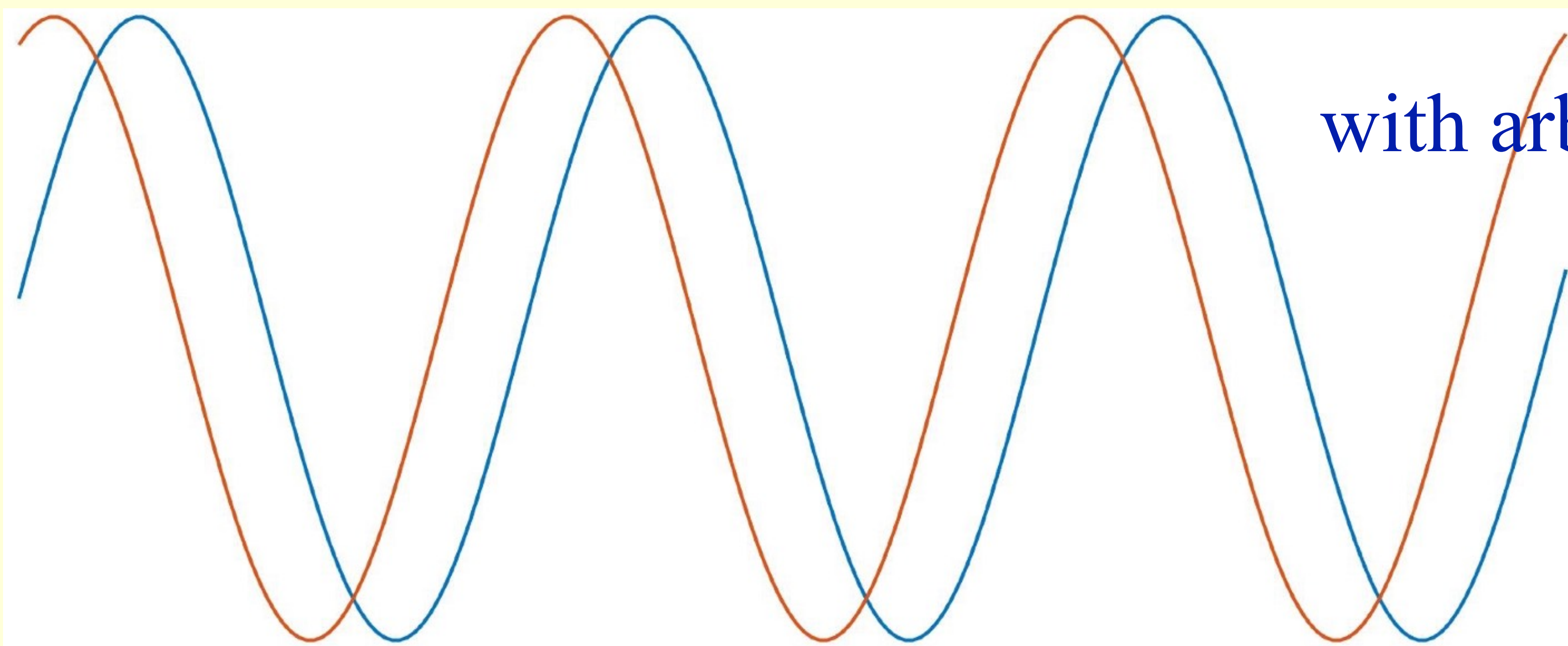
Synchronization: intermediate summary



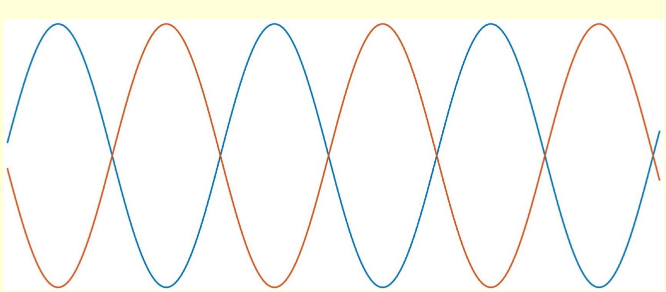
Synchronous state: <u>equal frequencies</u>



in-phase synchrony



with arbitrary phase shift



anti-phase synchrony

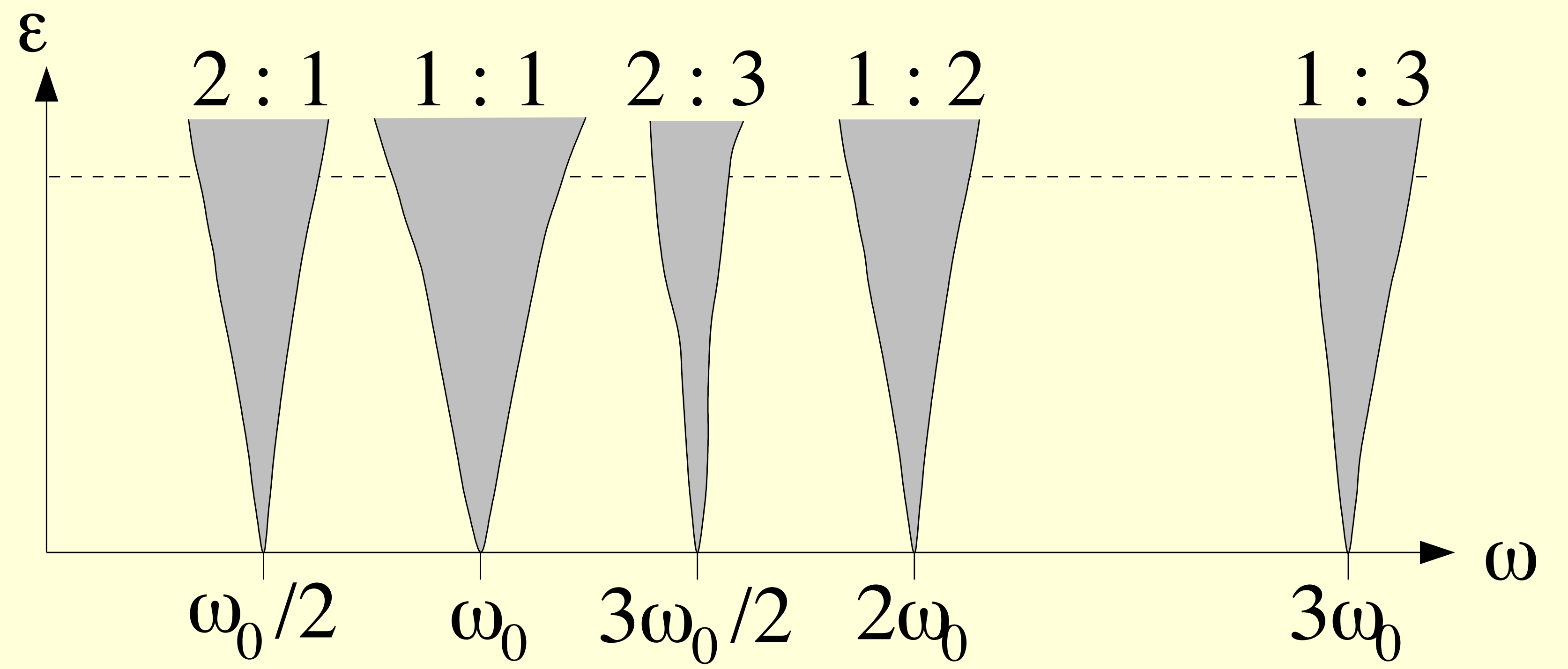
Frequency locking: intermediate summary

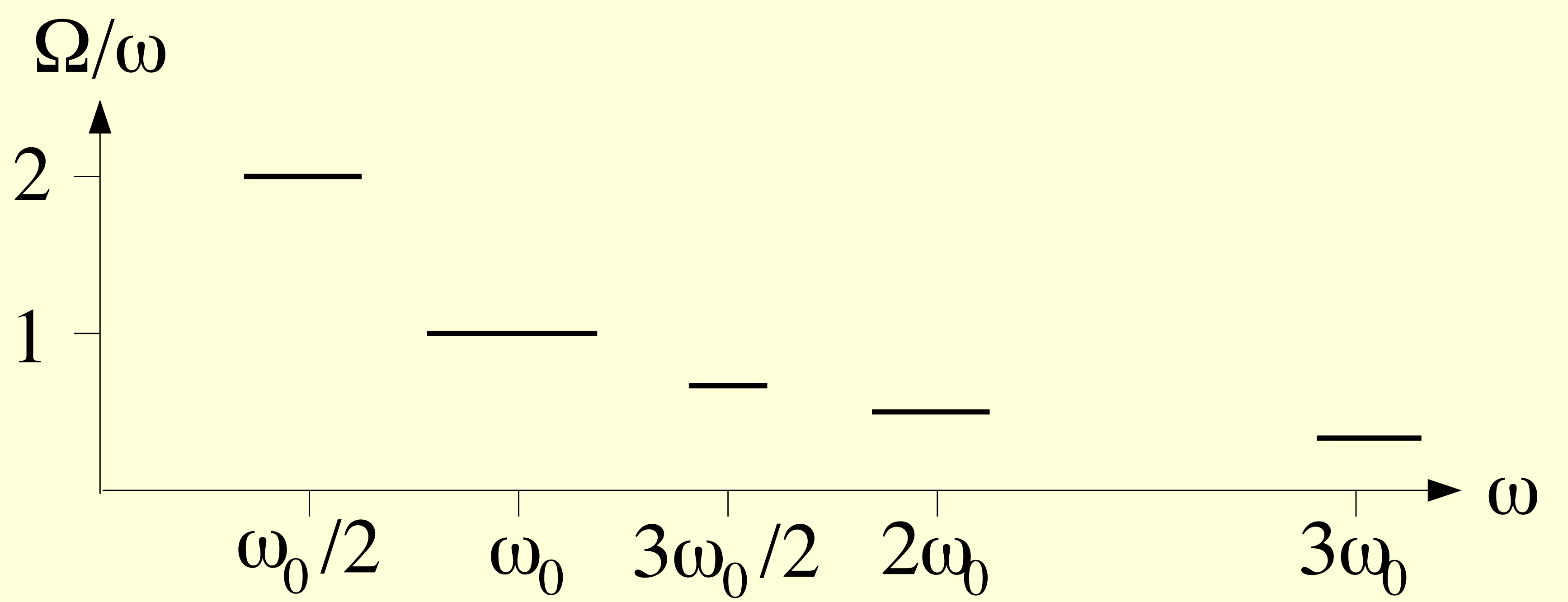
ω: forcing frequency

ε: forcing amplitude

ω_0: frequency of the autonomous system

Ω: frequency of the forced system





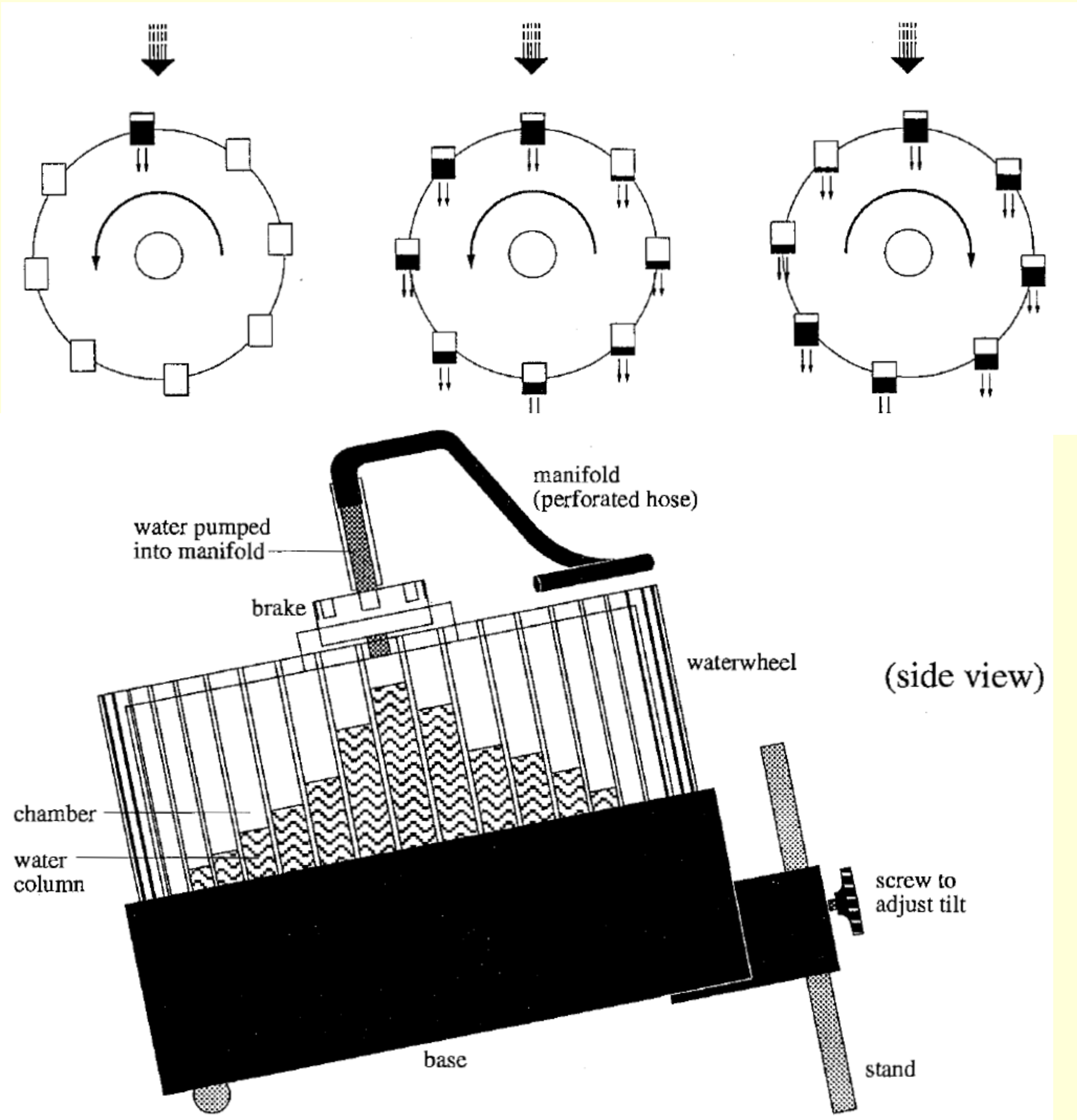
frequency locking:

$$n\Omega = m\omega$$

Same picture for mutual coupling of two systems

Chaotic oscillators

Waterwheel



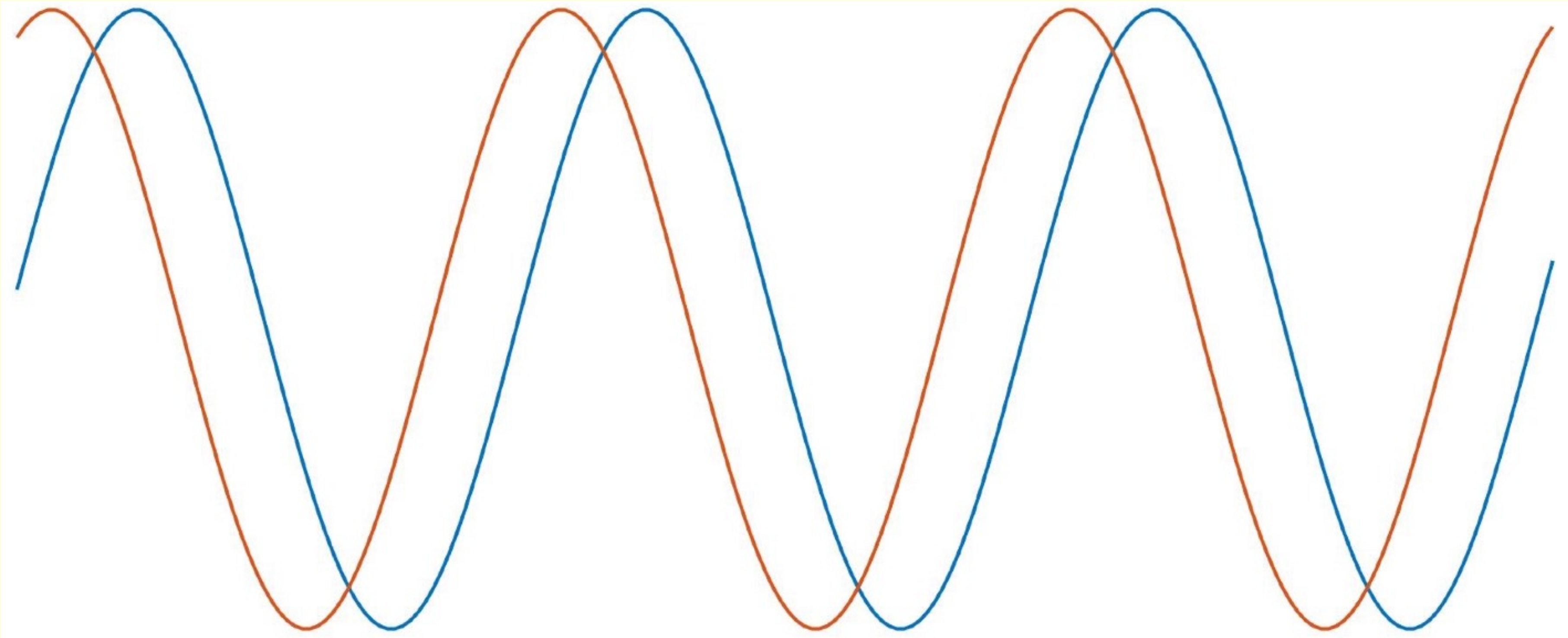
S. Strogatz, Nonlinear dynamics and chaos

Can chaotic systems synchronize? Yes!

Synchrony vs. simultaneity

Generally, synchrony does not imply simultaneity

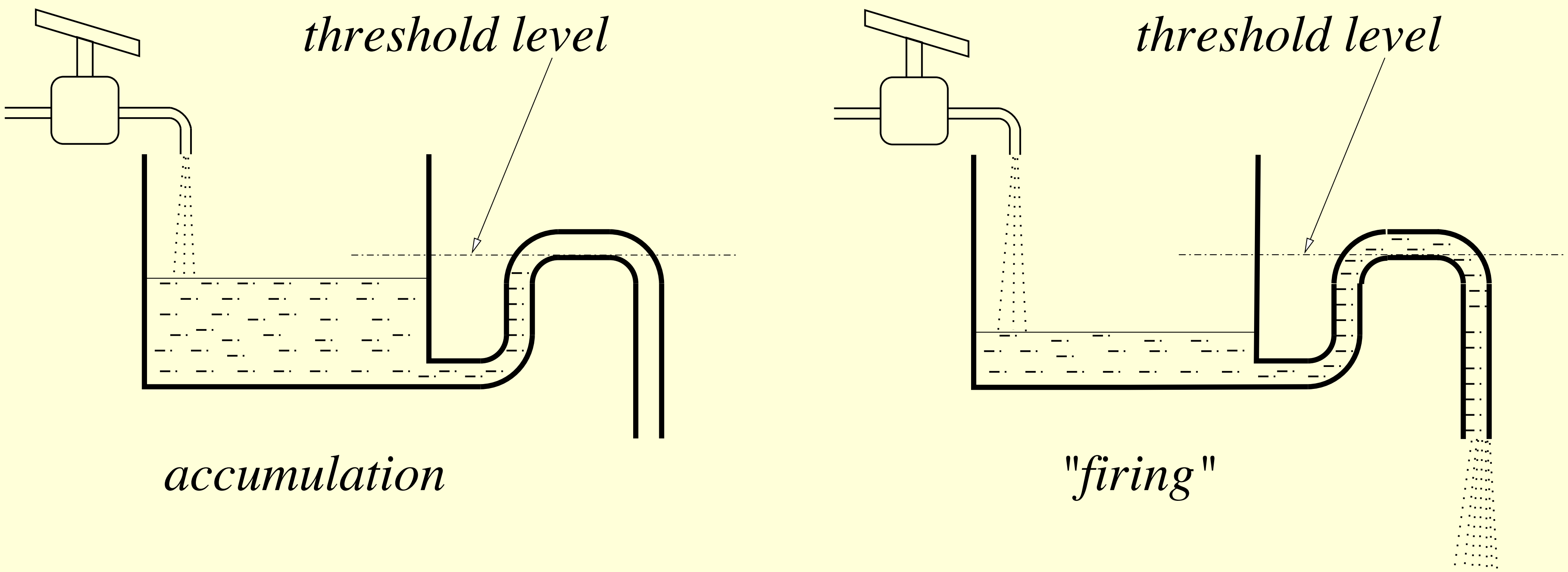
Recall synchronization with a phase shift:



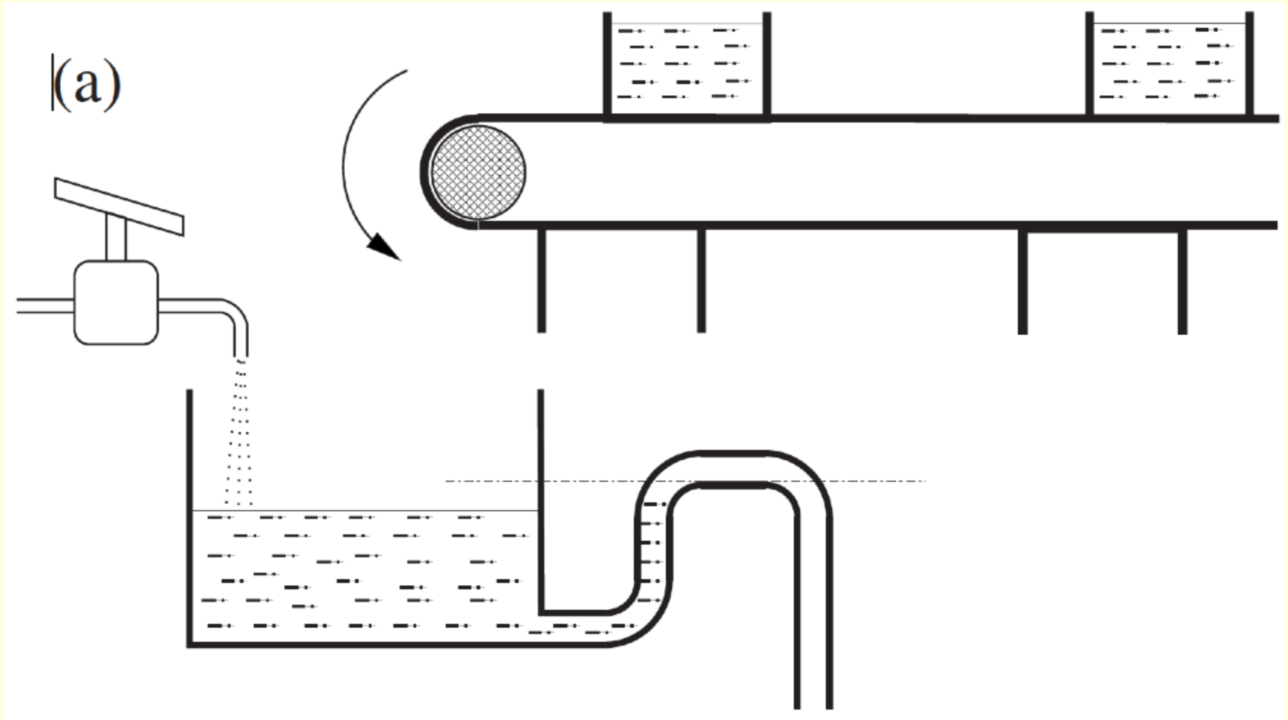
However, for some systems, e.g. for neurons, synchronization means simultaneous occurrence of events

Synchronization of integrate-and-fire systems

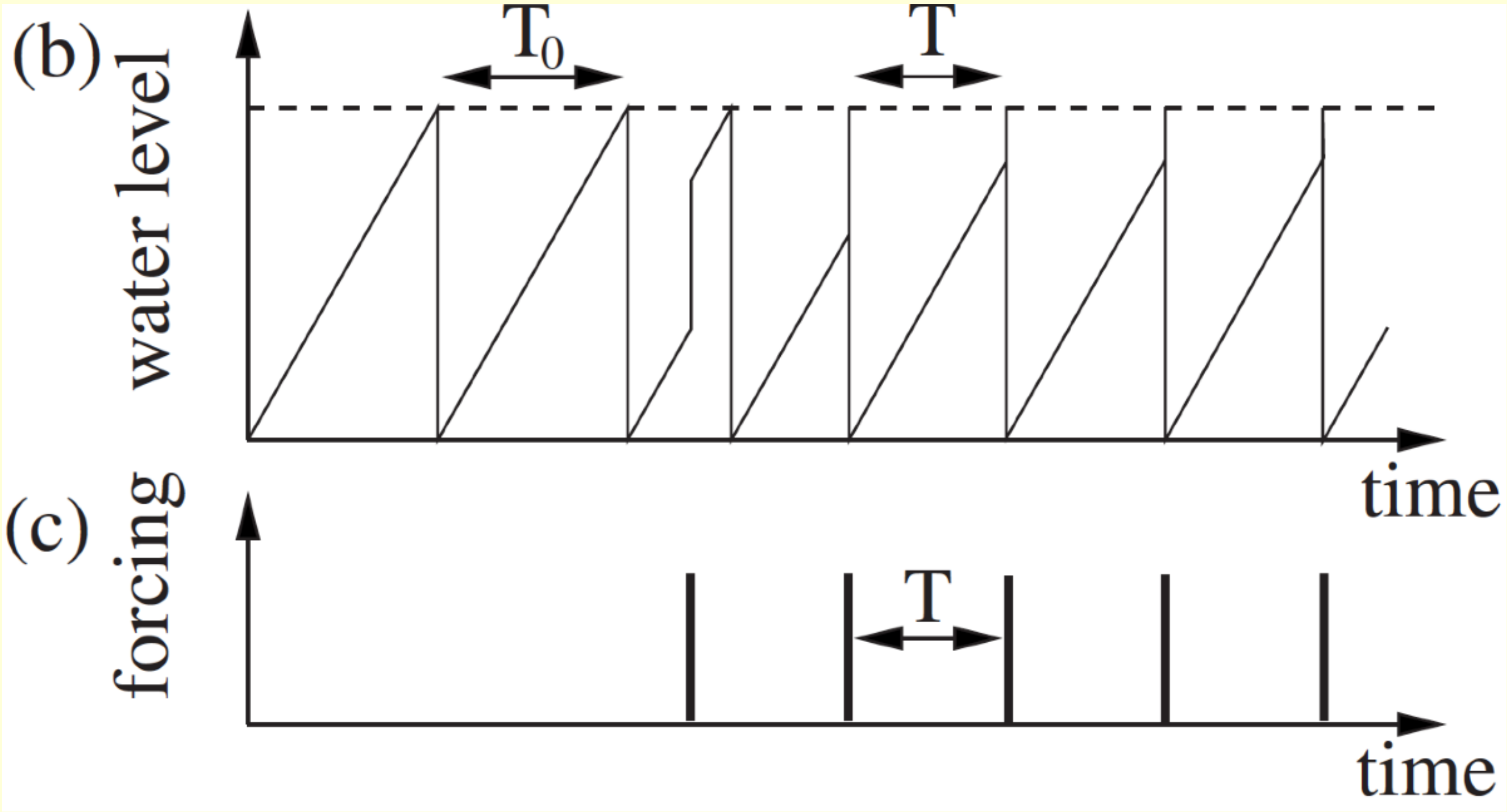
Recall the toy model



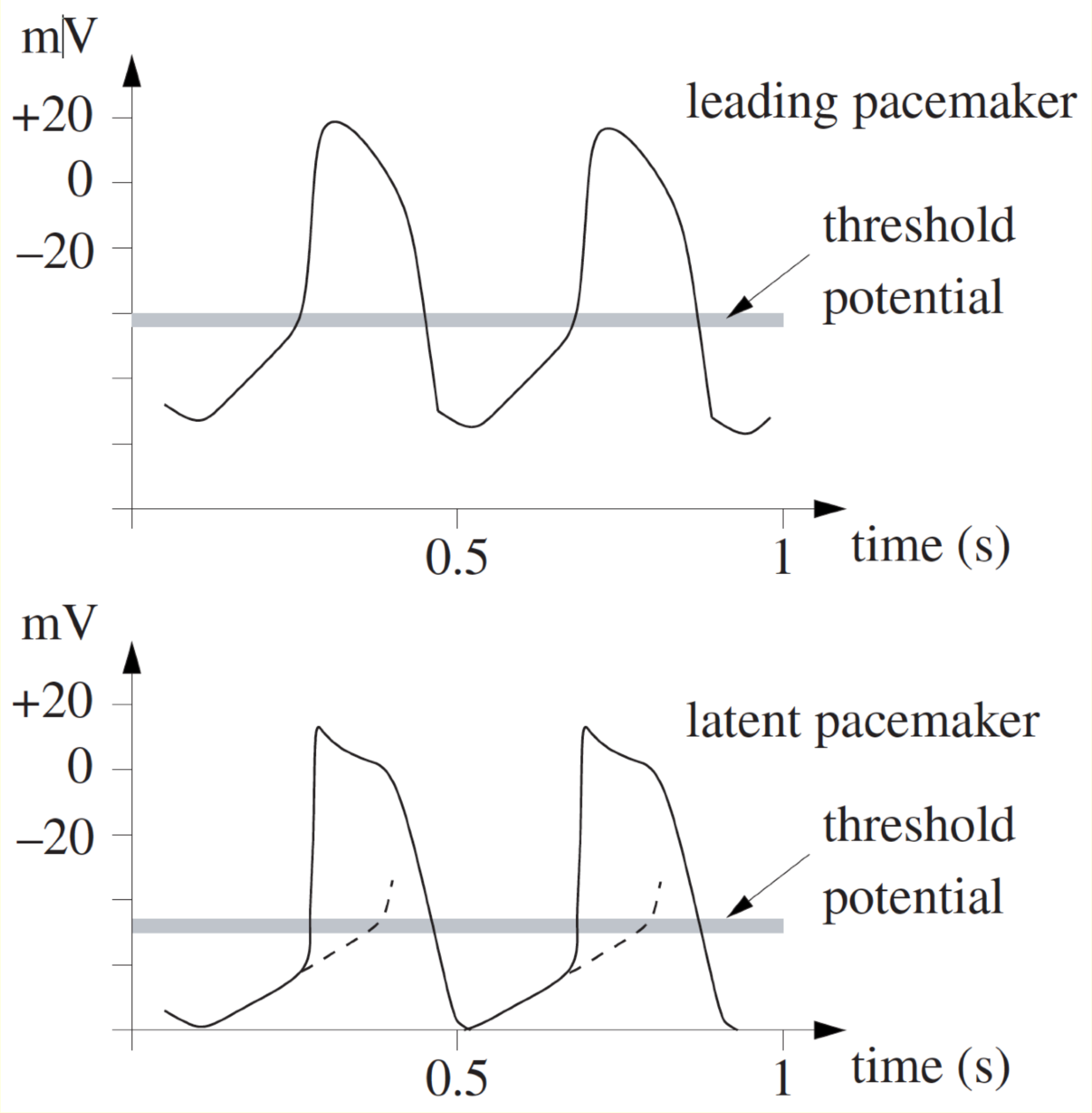
Forced integrate-and-fire systems



Here “synchronous” means “simultaneous”!



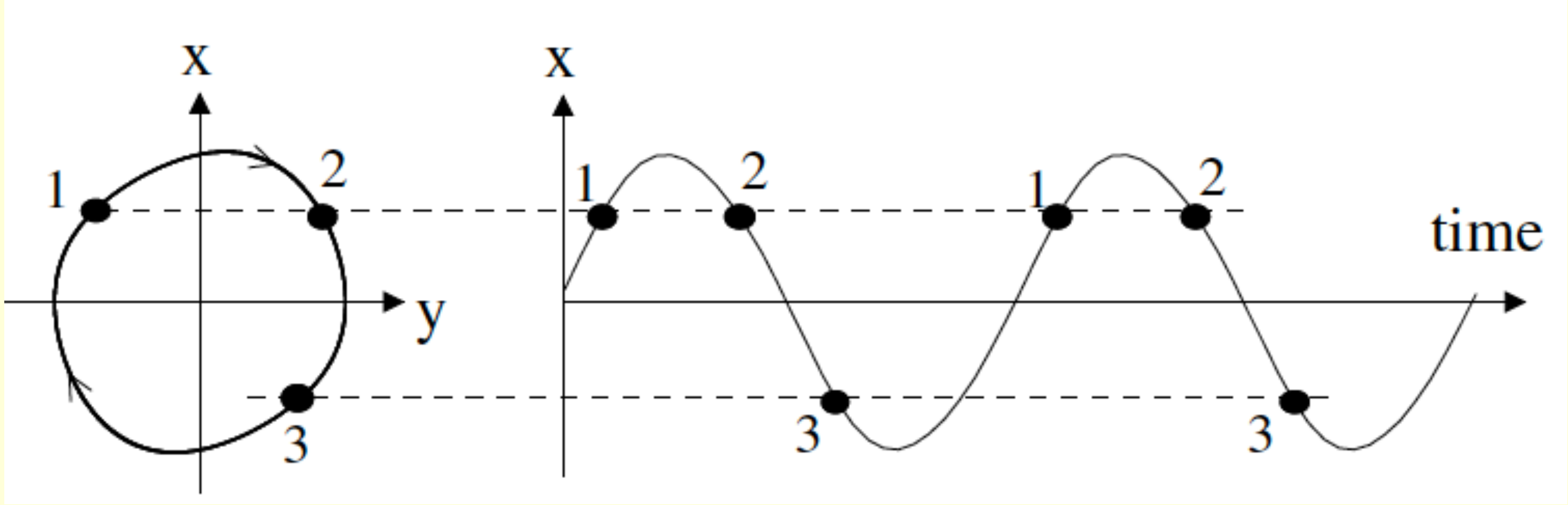
Example: cells of the sino-atrial node



From:
Dudel and Trautwein,1958,
Schmidt and Thews, 1983

Geometrical image of the periodic self-sustained oscillation: limit cycle

State of the clock is determined by the angle and velocity of the pendulum



Limit cycle

Consider general N-dimensional ($N \geq 2$) self-sustained oscillator

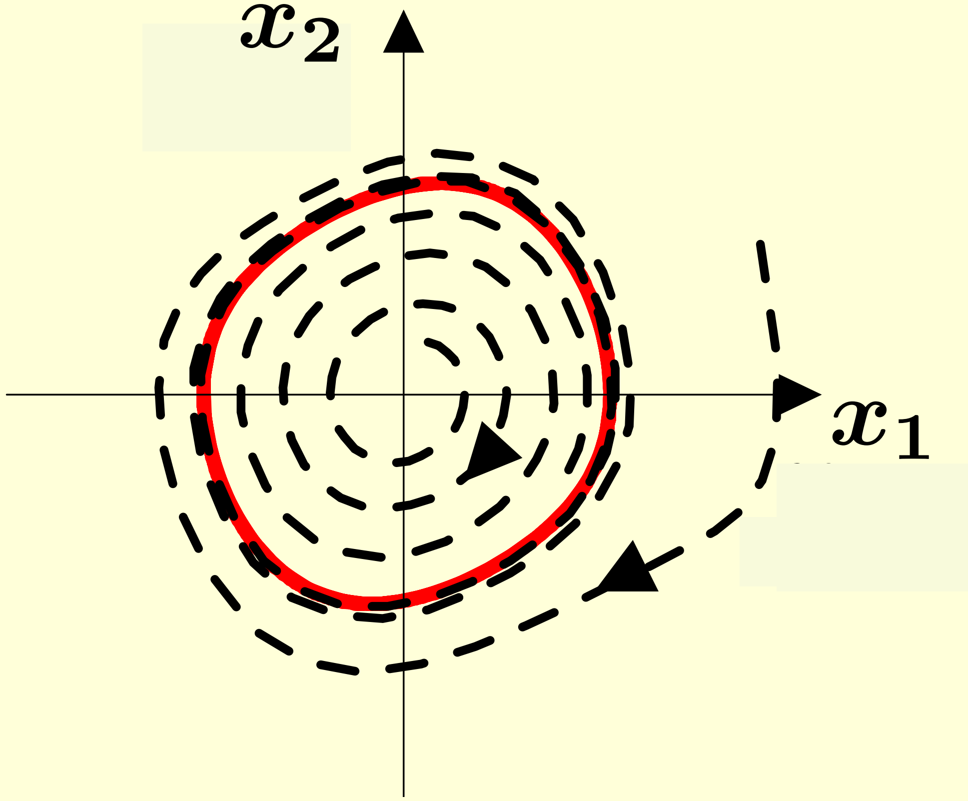
$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}) \ , \mathbf{x} = (x_1, x_2, \ldots, x_N)$$

Suppose it has a stable periodic solution

$$\mathbf{x}_0(t) = \mathbf{x}_0(t+T), \ T = 2\pi/\omega$$

In the **phase space** (the space of all variables $\mathbf{x}$) this solution is represented by an isolated closed attractive curve, called

limit cycle



Solvable model

Normal form equation for the Andronov-Hopf bifurcation

Stuart-Landau oscillator, Poincaré oscillator, Bautin oscillator, complex amplitude equation, ...

$$\dot{z} = (1 + i\omega)z - (1 + i\alpha)|z|^2 z$$

Polar coordinates: $z = Re^{i\varphi}$

$$\dot{R} = R(1 - R^2)$$

$$\dot{\varphi} = \omega - \alpha R^2$$

Limit cycle: $R = 1\,,\ \dot{\varphi} = \omega - \alpha$

Nonlinear, but solvable model!

Forced complex amplitude equation

$$\dot{z} = (1 + i\omega)z - (1 + i\alpha)|z|^2 z + \varepsilon e^{i\nu t}\ , \quad \alpha = 0\ , \varepsilon \ll 1$$

In polar coordinates:

$$\dot{R} = R(1 - R^2) + \varepsilon\cos(\nu t - \varphi)$$

$$\dot{\varphi} = \omega + \frac{\varepsilon}{R}\sin(\nu t - \varphi)$$

Approximate solution for a small deviation from the limit cycle:

$$R = 1 + \delta \longrightarrow \dot{\delta} \approx -2\delta + \varepsilon\cos(\nu t - \varphi)$$

$$\longrightarrow \delta \approx \frac{\varepsilon}{2}\cos(\nu t - \varphi)$$

$$R = 1 + \frac{\varepsilon}{2}\cos(\nu t - \varphi)$$

$$\dot{\varphi} = \omega + \varepsilon\sin(\nu t - \varphi)$$

Forced complex amplitude equation II

$$R = 1 + \frac{\varepsilon}{2}\cos(\nu t - \varphi)$$

$$\dot{\varphi} = \omega + \varepsilon \sin(\nu t - \varphi)$$

Amplitude dynamics: negligible variation of the amplitude

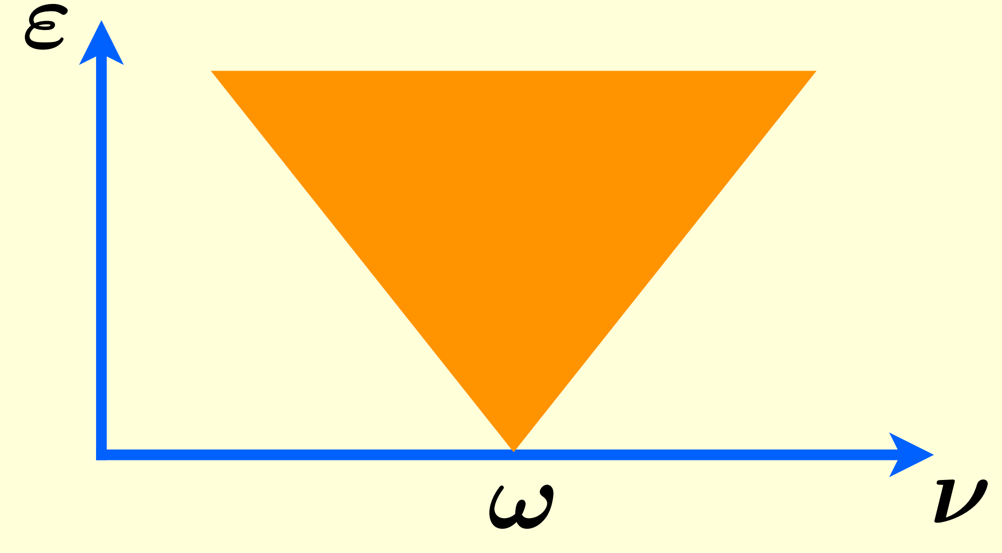
Phase dynamics: large deviation of the phase

Phase difference $\psi = \varphi - \nu t$

$$\dot{\psi} = \omega - \nu - \varepsilon \sin\psi$$

$$|\omega - \nu| \leq \varepsilon \Longrightarrow \psi = \text{const}$$

Synchronization!



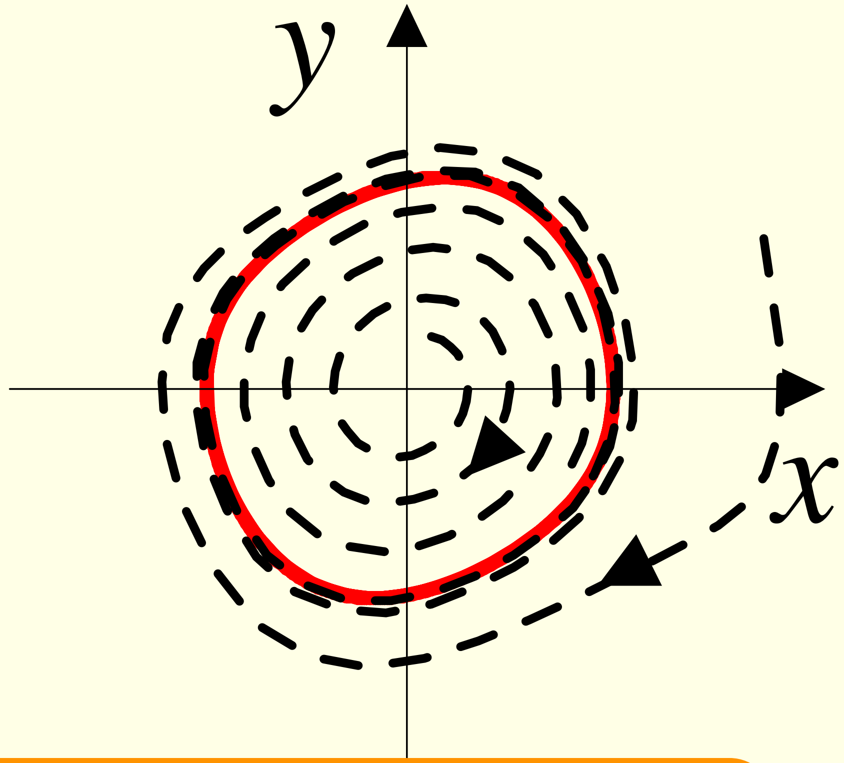
Phase is neutrally stable, amplitude is stable!

Consider two solutions on the limit cycle:

$$\frac{d\varphi}{dt} = \omega \quad \text{and} \quad \frac{d(\varphi + \Delta\varphi)}{dt} = \omega \quad \longrightarrow \quad \frac{d(\Delta\varphi)}{dt} = 0$$

Perturbation of the phase neither grows nor decays

Perturbation of the amplitude decays



Phase corresponds to the zero Lyapunov exponent

Amplitude corresponds to the negative Lyapunov exponent

Mathematical description of two coupled oscillators

Recall the general property of self-sustained oscillators:

1. amplitudes are stable
2. phases are free (neutrally stable)

Hence, for weak coupling, we consider the amplitudes as fixed and trace only the variation of phases

Indeed, exactly variation of phases determines adjustment of frequencies!

Uncoupled systems: $\dot{\phi}_1 = \omega_1 \ , \quad \dot{\phi}_2 = \omega_2$

Simplest model of phase dynamics of coupled systems:

$$\dot{\phi}_1 = \omega_1 + \varepsilon_1 \sin(\phi_2 - \phi_1)$$

$$\dot{\phi}_2 = \omega_2 + \varepsilon_2 \sin(\phi_1 - \phi_2)$$

Mathematical description of two coupled oscillators II

$$\dot{\phi}_1 = \omega_1 + \varepsilon_1 \sin(\phi_2 - \phi_1)$$

$$\dot{\phi}_2 = \omega_2 + \varepsilon_2 \sin(\phi_1 - \phi_2)$$

Phase difference: $\psi = \phi_1 - \phi_2$

$$\dot{\psi} = \omega_1 - \omega_2 - (\varepsilon_1 + \varepsilon_2) \sin \psi$$

Synchronous solution: $\dot{\psi} = 0 \quad \Rightarrow \quad \phi_1 - \phi_2 = \mathrm{const}$

$$\Rightarrow \quad \sin \psi = \frac{\omega_1 - \omega_2}{\varepsilon_1 + \varepsilon_2}$$

Synchronous solution exists if $|\omega_1 - \omega_2| \leq \varepsilon_1 + \varepsilon_2$

Frequency locking $\quad \Omega_{1,2} = \dot{\phi}_{1,2} \qquad \Omega_1 = \Omega_2$

Two coupled oscillators: general case

$$\dot{\phi}_1 = \omega_1 + \varepsilon_1 \sin(m\phi_2 - n\phi_1)$$

$$\dot{\phi}_2 = \omega_2 + \varepsilon_2 \sin(n\phi_1 - m\phi_2)$$

Phase difference: $\psi = n\phi_1 - m\phi_2$

$$\dot{\psi} = n\omega_1 - m\omega_2 - (n\varepsilon_1 + m\varepsilon_2)\sin\psi$$

Synchronous solution: $\dot{\psi} = 0 \quad \Rightarrow \quad \boxed{n\phi_1 - m\phi_2 = \text{const}}$

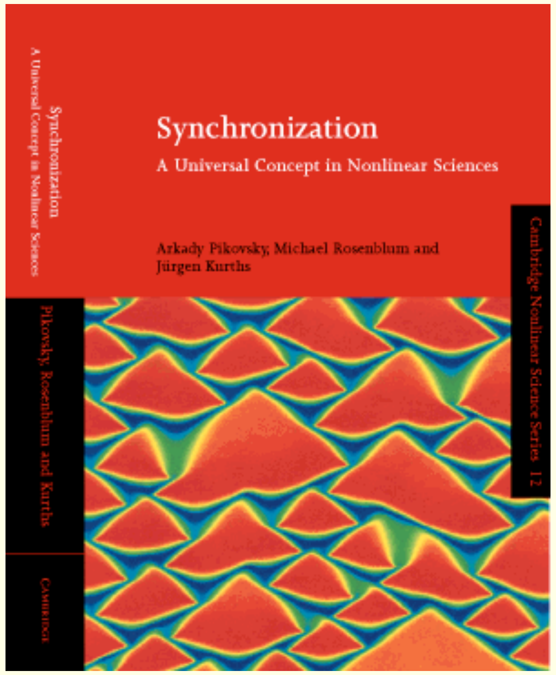
$$\Rightarrow \quad \sin\psi = \frac{n\omega_1 - m\omega_2}{n\varepsilon_1 + m\varepsilon_2}$$

Synchronous solution exists if $|n\omega_1 - m\omega_2| \le n\varepsilon_1 + m\varepsilon_2$

Frequency locking $\Omega_{1,2} = \dot{\phi}_{1,2}$ $\quad \boxed{n\Omega_1 = m\Omega_2}$

References

- A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization. A Universal Concept in Nonlinear Sciences, 2001



- S. Strogatz, Sync: The Emerging Science of Spontaneous Order, 2003